Theoretical cosmology

Cosmological perturbations

David Wands

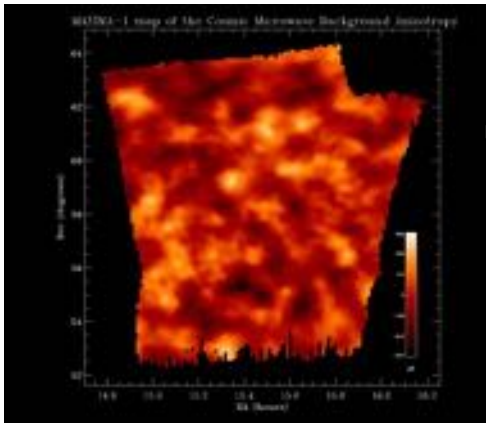
Institute of Cosmology and Gravitation
University of Portsmouth

5th Tah Poe School on Cosmology

Standard model of structure formation

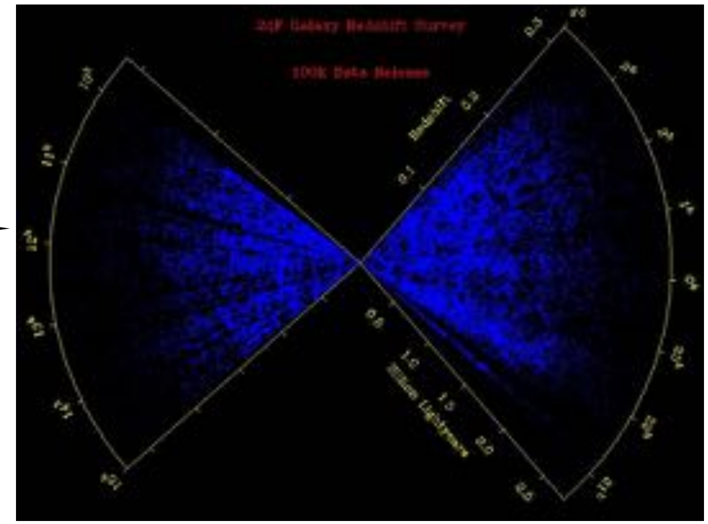
primordial perturbations

in cosmic microwave background



gravitational

instability



large-scale structure of our Universe

new observational data offers precision tests of

- cosmological parameters
- the nature of the primordial perturbations

Inflation: initial false vacuum state drives accelerated expansion
zero-point fluctuations yield spectrum of perturbations

References

- Malik and Wands, Phys Rep 475, 1 (2009), arXiv:0809.4944
- Bardeen, Phys Rev D22, 1882 (1980)
- Kodama and Sasaki, Prog Theor Phys Supp 78, 1 (1984)
- Bassett, Tsujikawa and Wands, Rev Mod Phys (2005), astro-ph/0507632

Outline

Homogeneous cosmology

Perturbation theory

Metric perturbations

Einstein equations

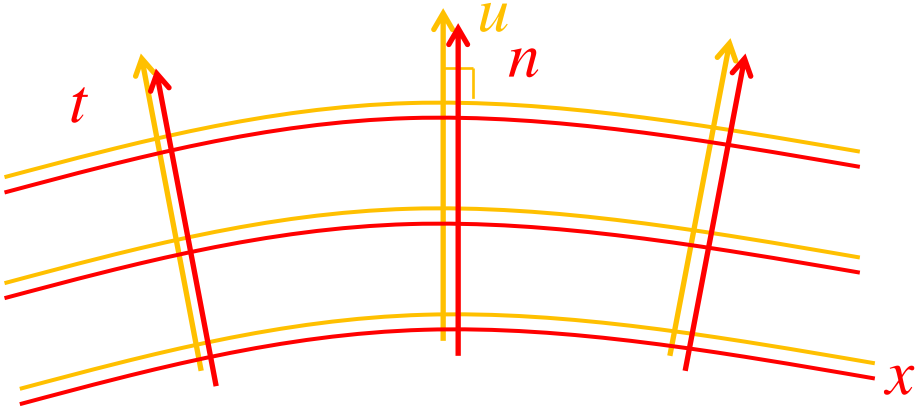
FLRW metric

- 4D spacetime split into 1+3
- Friedmann-Lemaitre-Robertson-Walker (FLRW) line element:

$$ds^2 = -c^2 dt^2 + a^2(t) dX^2$$

  - time + homogeneous and isotropic space
  - dynamical scale factor, $a(t)$, where $a_0 = 1$ today
  - maximally-symmetric 3-space, curvature K

$$dX^2 = \frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$



FRW cosmology

preferred coordinates for homogeneous and isotropic space

preferred space+time split in FRW cosmology breaks symmetry of Einstein's theory

FLRW metric

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$$ds^2 = a^2(\tau) \left[-d\tau^2 + dX^2 \right]$$

FLRW metric

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- Henceforth assume $K = 0$, flat space

Scalar perturbations

- Scalar quantity, e.g., density at fixed point P is invariant under change of coordinates
- Split into background (homogeneous) part and a perturbation (inhomogeneous):

$$\rho(t,\vec{x}) = \bar{\rho}(t) + \delta\rho(t,\vec{x})$$

  - expand perturbation order-by-order in a small parameter, ε:

$$\delta\rho(t,\vec{x}) = \varepsilon\delta_1\rho(t,\vec{x}) + \frac{1}{2}\varepsilon^2\delta_2\rho(t,\vec{x}) + \ldots$$

  - keep only terms at first order in $\varepsilon \Rightarrow$ *linear peturbations*

$$\delta\rho(t,\vec{x}) = \varepsilon\delta_1\rho(t,\vec{x})$$

Expanding equations order-by-order

- e.g., non-relativistic continuity equation for density $\rho(t,\vec{x})$

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0 \qquad (1)$$

expand density and velocity order-by-order

$$\begin{aligned}\rho(t,\vec{x}) &= \bar{\rho}(t)+\varepsilon\delta_1\rho(t,\vec{x})+\frac{1}{2}\varepsilon^2\delta_2\rho(t,\vec{x})+\dots\\ \vec{v}(t,\vec{x}) &= \varepsilon\delta_1\vec{v}(t,\vec{x})+\frac{1}{2}\varepsilon^2\delta_2\vec{v}(t,\vec{x})+\dots\end{aligned}$$

substitute into Eq. (1)

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\bar{\rho}+\varepsilon\delta_1\rho+\frac{1}{2}\varepsilon^2\delta_2\rho+\dots\right)\\ &+\vec{\nabla}\cdot\left[\left(\bar{\rho}+\varepsilon\delta_1\rho+\frac{1}{2}\varepsilon^2\delta_2\rho+\dots\right)\right.\\ &\left.\times\left(\varepsilon\delta_1\vec{v}+\frac{1}{2}\varepsilon^2\delta_2\vec{v}+\dots\right)\right]=0\end{aligned}$$

Perturbation equations order-by-order

- collect terms order-by-order in ε

$$
\begin{aligned}
&\frac{\partial}{\partial t}\bar{\rho} \\
&+\varepsilon\left\{\frac{\partial}{\partial t}\delta_1\rho + \vec{\nabla}\cdot(\bar{\rho}\delta_1\vec{v})\right\} \\
&+\frac{1}{2}\varepsilon^2\left\{\frac{\partial}{\partial t}\delta_2\rho + \vec{\nabla}\cdot(\bar{\rho}\delta_2\vec{v} + 2\delta_1\rho\delta_1\vec{v})\right\} + \ldots = 0
\end{aligned}
$$

- solve order-by-order in ε

$$
\begin{aligned}
\frac{\partial}{\partial t}\bar{\rho} &= 0 \qquad \Rightarrow \qquad \bar{\rho} = C \\
\frac{\partial}{\partial t}\delta_1\rho + C\vec{\nabla}\cdot\delta_1\vec{v} &= 0 \\
\frac{\partial}{\partial t}\delta_2\rho + C\vec{\nabla}\cdot\delta_2\vec{v} &= -2C\vec{\nabla}\cdot(\delta_1\rho\,\delta_1\vec{v})
\end{aligned}
$$

Fourier transform

- Field in real space is an integral over Fourier modes:

$$\delta\rho(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\,\delta\rho_{\vec{k}}(t)\,e^{i\vec{k}.\vec{x}}$$

- Fourier modes are eigenfunctions of the spatial Laplacian:

$$\nabla^2\left(e^{i\vec{k}.\vec{x}}\right) = -k^2 e^{i\vec{k}.\vec{x}}$$

which provide a complete orthonormal basis:

$$\int d^3x\, e^{i\vec{k}_1.\vec{x}}\, e^{i\vec{k}_2.\vec{x}} = (2\pi)^3\delta^{(3)}\left(\vec{k}_1 - \vec{k}_2\right)$$

- Coefficient in Fourier space is integral over real space:

$$\delta\rho_{\vec{k}}(t) = \int d^3x\,\delta\rho(t,\vec{x})\,e^{-i\vec{k}.\vec{x}}$$

Statistical distribution

- *theory*
  describes properties of distribution = ensemble, assumed isotropic
  ($\langle \ldots \rangle$ = average over all possible realisations)
- *observations*
  describe one realisation from the distribution

Power spectrum

- defined by the correlation of two modes in Fourier space:

$$\langle \delta\rho_{\vec{k}_1}\delta\rho_{\vec{k}_2}\rangle = (2\pi)^3\, P_\rho(k_1)\delta^3\left(\vec{k}_1+\vec{k}_2\right)$$

  note: $P_\rho(k)$ only a function of wavenumber k, not wavevector $\vec{k}$, for an isotropic distribution
- Variance in real space: *(exercise for reader!)*

$$\begin{aligned}\langle\delta\rho^2(\vec{x})\rangle &= \langle\int \frac{d^3\vec{k}_1 d^3\vec{k}_2}{(2\pi)^6}\delta\rho_{\vec{k}_1}\delta\rho_{\vec{k}_2}e^{i\left(\vec{k}_1+\vec{k}_2\right)\cdot\vec{x}}\rangle \\ &= \int \frac{d^3\vec{k}_1 d^3\vec{k}_2}{(2\pi)^6}\langle\delta\rho_{\vec{k}_1}\delta\rho_{\vec{k}_2}\rangle e^{i\left(\vec{k}_1+\vec{k}_2\right)\cdot\vec{x}} \\ &= \int \frac{d^3\vec{k}_1}{(2\pi)^3}P_\rho(k_1) = \int d\ln k_1\,\mathcal{P}_\rho(k_1)\end{aligned}$$

- dimensionless power spectrum per log k:

$$\mathcal{P}_\rho(k) = \frac{4\pi k^3}{(2\pi)^3}P_\rho(k)$$

Higher-order statistics

- Bispectrum

$$\langle \delta\rho_{\vec{k}_1} \delta\rho_{\vec{k}_2} \delta\rho_{\vec{k}_3} \rangle = (2\pi)^3 \, B_\rho(k_1, k_{2,3}\,)\delta^3\left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3\right)$$

  Bispectrum is zero for *Gaussian* perturbations (and for all odd moments)

- We will take *first-order* perturbations to be *Gaussian*:

$$\langle \delta_1\rho_{\vec{k}_1} \delta_1\rho_{\vec{k}_2} \delta_1\rho_{\vec{k}_3} \rangle = 0$$

  Second- and higher-order perturbations are *non-Gaussian*.

$$\langle \delta_2\rho_{\vec{k}_1} \delta_1\rho_{\vec{k}_2} \delta_1\rho_{\vec{k}_3} \rangle \neq 0$$

Vector perturbations

- decompose any 3-vector: $\vec{V} = \vec{\nabla} V^{(s)} + \vec{V}^{(v)}$
  - *scalar* (longitudinal/potential) flow: $\vec{\nabla} \times \vec{\nabla} V^{(s)} = 0$
  - *vector* (transverse/divergence-free) flow: $\vec{\nabla} \cdot \vec{V}^{(v)} = 0$

- Fourier transform
  - *scalar*

$$V^{(s)}(t, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} V_{\vec{k}}^{(s)}(t)\, e^{i\vec{k}.\vec{x}}$$

  - *vector*

$$\vec{V}^{(v)}(t, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} \left\{ V_{\vec{k}}^{(v)}(t) \vec{e}_{\vec{k}} + \tilde{V}_{\vec{k}}^{(v)}(t) \vec{\tilde{e}}_{\vec{k}} \right\} e^{i\vec{k}.\vec{x}}$$

where $\vec{e}_{\vec{k}}$ and $\vec{\tilde{e}}_{\vec{k}}$ are orthonormal *polarisation* vectors:

$$\vec{e}_{\vec{k}} \cdot \vec{e}_{\vec{k}} = \vec{\tilde{e}}_{\vec{k}} \cdot \vec{\tilde{e}}_{\vec{k}} = 1\,, \quad \vec{e}_{\vec{k}} \cdot \vec{\tilde{e}}_{\vec{k}} = 0$$

transverse to wavevector $\vec{k}$:

$$\vec{k} \cdot \vec{e}_{\vec{k}} = \vec{k} \cdot \vec{\tilde{e}}_{\vec{k}} = 0$$

Vector perturbations

putting it all together

$$\vec{V}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left\{ i\vec{k}\, V^{(s)}_{\vec{k}}(t) + \vec{e}_{\vec{k}} V^{(v)}_{\vec{k}}(t) + \tilde{\vec{e}}_{\vec{k}} \tilde{V}^{(v)}_{\vec{k}}(t) \right\} e^{i\vec{k}.\vec{x}}$$

Tensor perturbations

- decompose any 3-tensor:
  $T_{ij} = \delta_{ij} C + \nabla_i \nabla_j S + (1/2) \left(\nabla_i V_j + \nabla_j V_i \right) + h_{ij}$
  - *scalars* C and S are longitudinal/potential
  - *vector* V_i is transverse: $\nabla^i V_i = 0$
  - *tensor* h_{ij} is transverse and trace-free:

$$\nabla^i h_{ij} = \nabla^j h_{ij} = 0 \, , \qquad h^i_i = 0$$

Tensor perturbations

- Fourier transform:

$$h_{ij}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left\{ h^{(+)}_{\vec{k}}(t) q^{(+)}_{\vec{k}\,ij} + h^{(\times)}_{\vec{k}}(t) q^{(\times)}_{\vec{k}\,ij} \right\} e^{i\vec{k}.\vec{x}}$$

where *polarisation* tensors

$$q^{(+)}_{\vec{k}\,ij} = \frac{1}{\sqrt{2}} \left(e_{\vec{k}\,i} e_{\vec{k}\,j} - \tilde{e}_{\vec{k}\,i} \tilde{e}_{\vec{k}\,j} \right)$$
$$q^{(\times)}_{\vec{k}\,ij} = \frac{1}{\sqrt{2}} \left(e_{\vec{k}\,i} \tilde{e}_{\vec{k}\,j} + \tilde{e}_{\vec{k}\,i} e_{\vec{k}\,j} \right)$$

and $e_{\vec{k}\,i}$ and $\tilde{e}_{\vec{k}\,i}$ are orthonormal, transverse vectors, such that *(exercise for reader!)*

$$q^{(+)\,ij}_{\vec{k}} q^{(+)}_{\vec{k}\,ij} = q^{(\times)\,ij}_{\vec{k}} q^{(\times)}_{\vec{k}\,ij} = 1 \,, \quad q^{(+)\,ij}_{\vec{k}} q^{(\times)}_{\vec{k}\,ij} = 0$$

tracefree $q^{(+)i}_{\vec{k}\;\;i} = q^{(\times)i}_{\vec{k}\;\;i} = 0$ and transverse to $\vec{k}$:

$$k^i q^{(+)}_{\vec{k}\,ij} = k^i q^{(\times)}_{\vec{k}\,ij} = 0$$

Metric perturbations

- Split metric into spatially-flat FLRW background and inhomogeneous perturbation:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \,.$$

- Background:

$$\bar{g}_{00} = a^2 \,, \quad \bar{g}_{0i} = 0 \,, \quad \bar{g}_{ij} = a^2 \delta_{ij} \tag{2}$$

- Perturbation:

$$\begin{aligned} \delta g_{00} &= 2a^2 A \\ \delta g_{0i} &= a^2 \left(\nabla_i B - S_i \right) \\ \delta g_{ij} &= a^2 \left(2C\delta_{ij} + 2\nabla_i \nabla_j E + \nabla_i F_j + \nabla_j F_i + h_{ij} \right) \end{aligned}$$

  - 4 scalars: A, B, C, E
  - 2 vectors: S_i, F_i
  - 1 tensor: h_{ij}

Metric perturbations

- Perturbed line-element including only scalar perturbations:

$$
\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\
&= a^2(\tau)\left\{-(1+2A)d\tau^2 + 2(\partial_i B)dx^i d\tau \right. \\
&\qquad \left. + \left[(1+2C)\delta_{ij} + 2(\partial_{ij}E)\right]\right\} dx^i dx^j
\end{aligned}
$$

  where four scalar perturbations are
  - A = lapse perturbation
  - $\partial_i B = \partial B/\partial x^i$ = shift perturbation
  - C = spatial curvature perturbation
  - $\partial_{ij} E = \partial^2 E/\partial x^i \partial x^j$ = off-diagonal spatial perturbation

Geometrical interpretation

- Temporal gauge (time-slicing) in 4D spacetime defines a hypersurface orthogonal 4-vector field:

$$N_\mu \propto \frac{\partial \tau}{\partial x^\mu}$$

normalise such that $N_\mu N^\mu = -1$.

  - intrinsic curvature of constant τ hypersurfaces:

$${}^{(3)}R = -\frac{4}{a^2}\nabla^2 C$$

  - expansion of constant τ hypersurfaces:

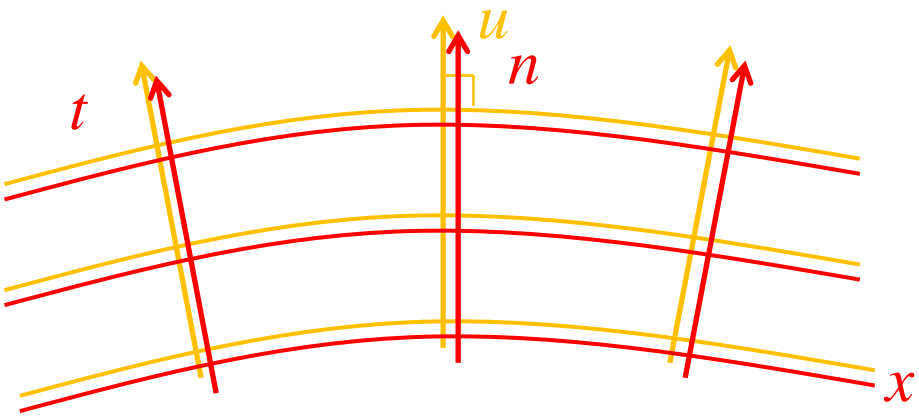
$$\theta = \frac{3}{a}\left(\frac{a'}{a}(1-A) + C' + \frac{1}{3}\nabla^2\sigma\right)$$

  - shear:

$$\sigma_{ij} = \left(\nabla_i\nabla_j - \frac{1}{3}\nabla^2\right)\sigma\,, \qquad \sigma = E' - B$$

  - acceleration:

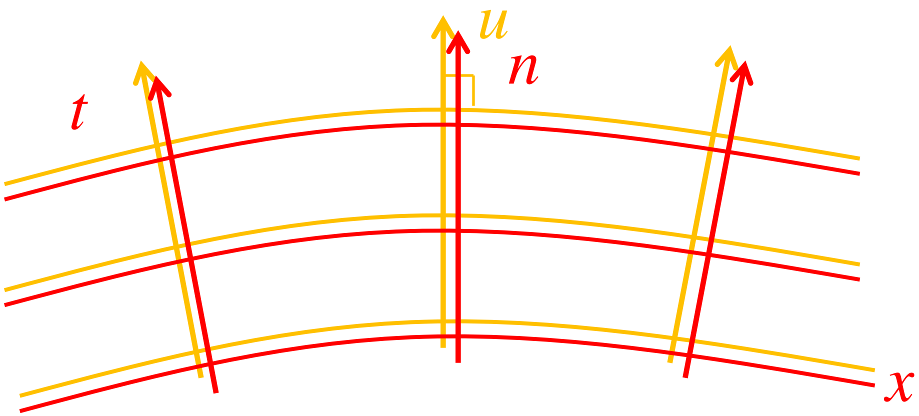
$$a_i = \nabla_i A$$



FRW cosmology

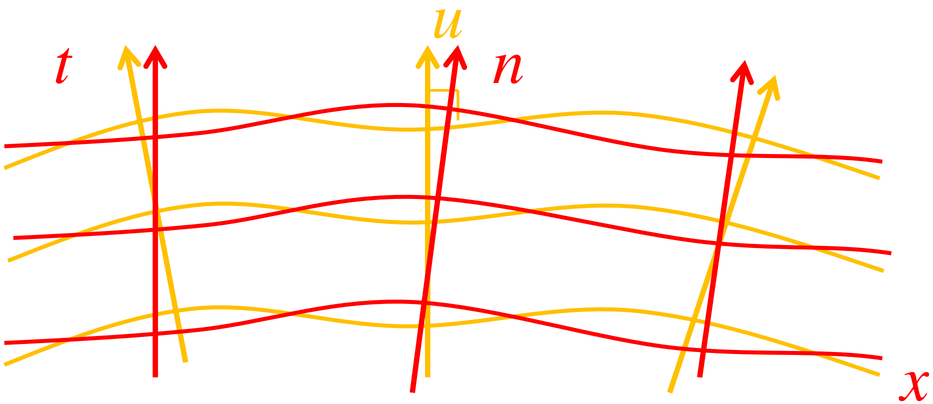
preferred coordinates for homogeneous and isotropic space

preferred space+time split in FRW cosmology breaks symmetry of Einstein's theory



FRW cosmology

no unique choice of time (slicing) and space coordinates (threading) in an inhomogeneous universe



FRW cosmology
+ perturbations

arbitrary gauge (t,x)

gauge problem: find different perturbations in different gauges

Gauge dependence

- Scalar quantity, e.g., density, $\rho|_P$, at given point P is invariant
$$\rho(\tau,\vec{x})|_P = \tilde{\rho}(\tilde{\tau},\tilde{\vec{x}})|_P$$
under first-order (scalar) change of coordinates:
$$\begin{aligned}\tilde{\tau} &= \tau + \delta\tau(\tau,\vec{x}) \\ \tilde{\vec{x}} &= \vec{x} + \vec{\nabla}\delta x(\tau,\vec{x})\end{aligned}$$
- but background-perturbation split is gauge-dependent
$$\begin{aligned}\rho_0(\tau) + \delta\rho|_P &= \rho_0(\tilde{\tau}) + \widetilde{\delta\rho}|_P \\ \Rightarrow \widetilde{\delta\rho}|_P &= \delta\rho|_P + \rho_0(\tau) - \rho_0(\tilde{\tau}) \\ &= \delta\rho|_P - \rho_0'\delta\tau \end{aligned} \tag{3}$$

Linear gauge transformation rules

- Coordinate change:

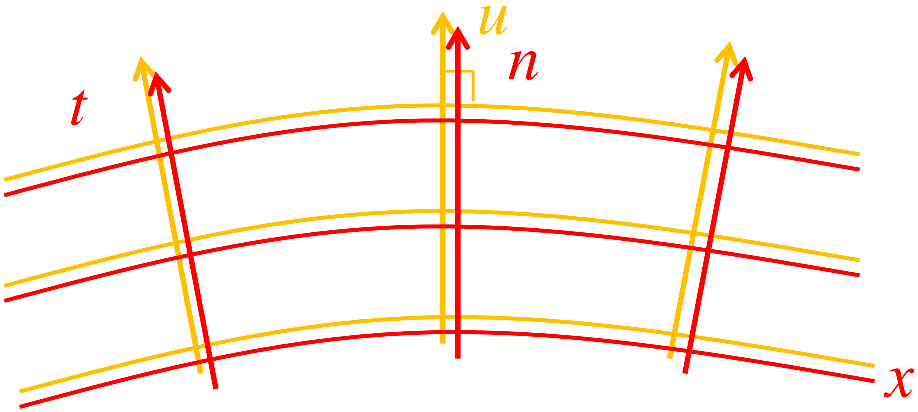
$$\begin{aligned} \text{time-slicing:} \quad \tilde{\tau} &\rightarrow \tau + \delta\tau(\tau, \vec{x}) \\ \text{spatial-threading:} \quad \tilde{\vec{x}} &\rightarrow \vec{x} + \vec{\nabla}\delta x(\tau, \vec{x}) \end{aligned}$$

- Gauge transformations:

$$\begin{aligned} \text{density:} \quad \widetilde{\delta\rho} &= \delta\rho - \rho'\delta\tau \\ \text{pressure:} \quad \widetilde{\delta P} &= \delta P - P'\delta\tau \\ \text{velocity:} \quad \tilde{v}^i &= v^i + \partial^i \delta x \end{aligned} \tag{4}$$

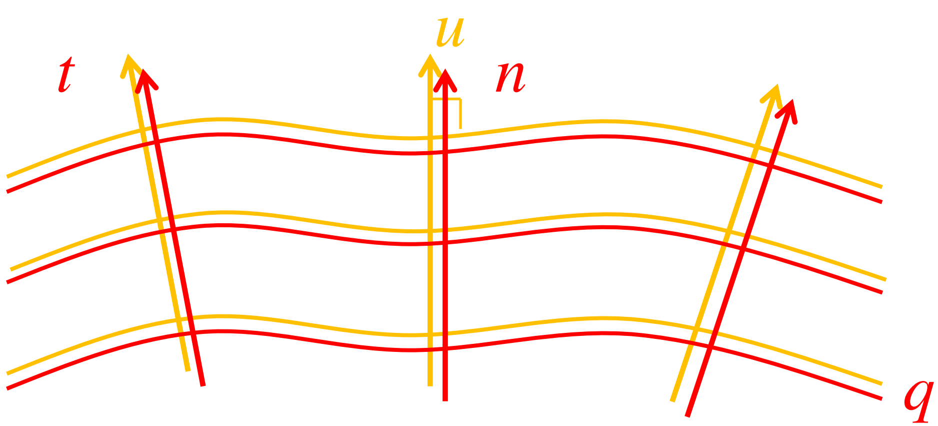
including three metric transformations independent of spatial-threading:

$$\begin{aligned} \text{lapse:} \quad \tilde{A} &= A - \frac{a'}{a}\delta\tau - \delta\tau' \\ \text{curvature:} \quad \tilde{C} &= C - \frac{a'}{a}\delta\tau \\ \text{shear:} \quad \tilde{\sigma} = \tilde{E}' - \tilde{B} &= \sigma - \delta\tau \end{aligned} \tag{5}$$



FRW cosmology

synchronous+comoving with pressureless cold dark matter
time-slicing orthogonal to comoving worldlines



FRW cosmology
+ perturbations

comoving-Lagrangian coordinates (t,q)

Conformal Newtonian/Longitudinal gauge

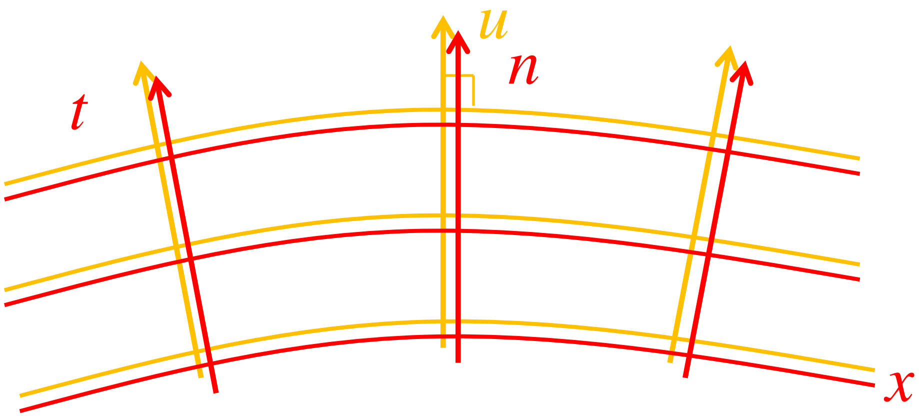
- pick a gauge to completely fix the coordinates
- for example: *longitudinal gauge (zero-shear time-slices)*:
  - set $\sigma \to \tilde{\sigma} = 0$ which requires a transform $\delta\tau = \sigma$
  - we then have

$$\text{density:} \quad \delta \equiv \frac{\delta\rho}{\rho} \quad \to \quad \widetilde{\delta} = \delta - \frac{\rho'}{\rho}\sigma \tag{6}$$

including two gauge-invariant metric perturbations:

$$\text{lapse:} \quad A \quad \to \quad \Psi \equiv A - \frac{a'}{a}\sigma - \sigma'$$

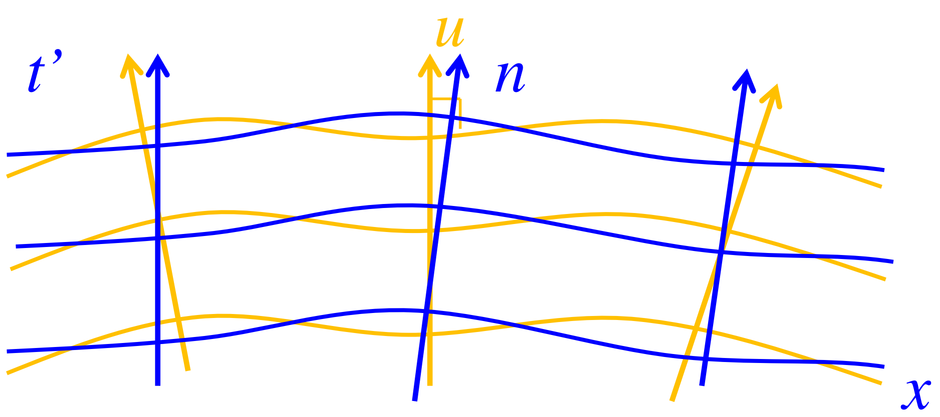
$$\text{curvature:} \quad C \quad \to \quad \Phi \equiv C - \frac{a'}{a}\sigma \tag{7}$$



FRW cosmology

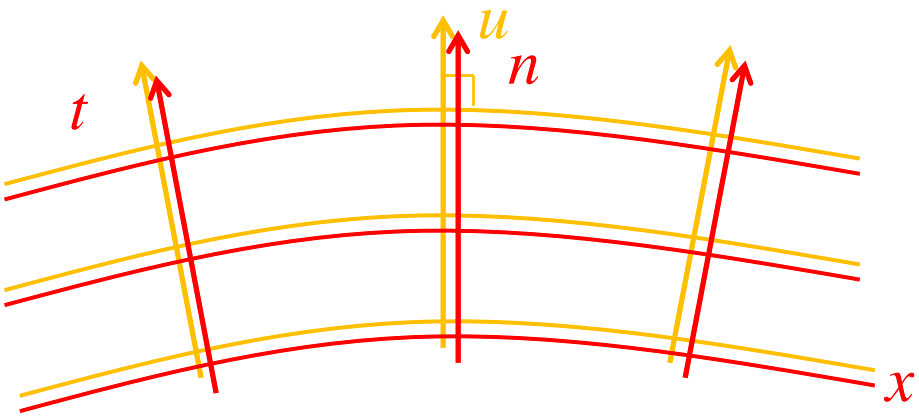
Poisson = conformal Newtonian = longitudinal gauge

*hypersurface-orthogonal 4-vector field **n** is shear-free*



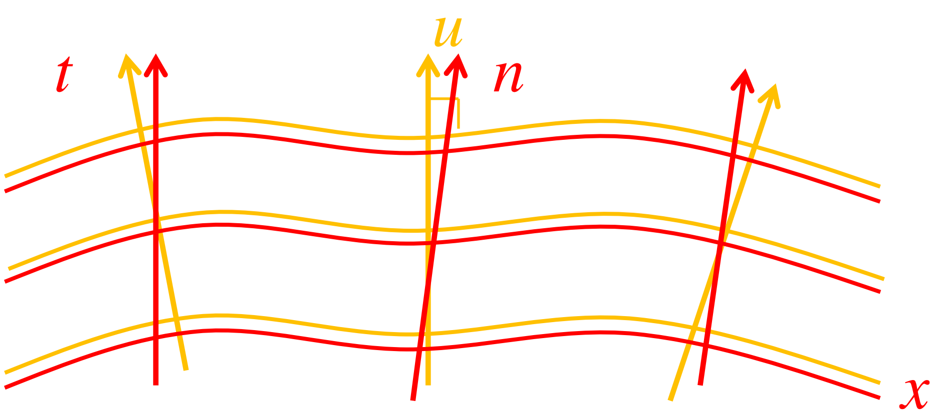
FRW cosmology
+ perturbations

Poisson gauge coordinates *(t',x)*



FRW cosmology

time-slicing orthogonal to comoving worldlines
spatial threading is same as Poisson gauge (Eulerian, not Lagrangian)



FRW cosmology
+ perturbations

total-matter coordinates *(t,x)*

Standard Newtonian+Gaussian initial fields

Gaussian primordial metric fluctuations $\zeta(x)$ from inflation + linear Einstein-Boltzmann code (e.g., CMBfast, CAMB, CLASS)

Gaussian initial Newtonian potential $\Phi = (3/5)\zeta$

Gaussian initial matter density using Poisson equation

$$\nabla^2\Phi = 4\pi G a^2 \bar{\rho}\delta$$

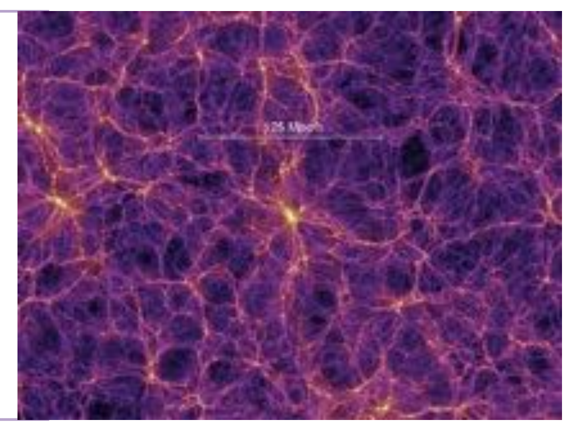
Gaussian initial displacement $\vec{\nabla}.\vec{\Psi} = -\delta$

Newtonian N-body simulations

$$\nabla^2\Phi = 4\pi G a^2 \bar{\rho}\delta$$

$$\dot{\delta} + \vec{\nabla}.((1+\delta))\vec{v} = 0$$

$$\dot{\vec{v}} + \mathcal{H}\vec{v} + (\vec{v}.\vec{\nabla})\vec{v} = -\vec{\nabla}\Phi$$



Uniform-density gauge

- pick a gauge to completely fix the coordinates
- for example: *uniform-density time-slices*:
  - set $\delta\rho \to \widetilde{\delta\rho} = 0$ which requires a transform $\delta\tau = \delta\rho/\rho'$
  - we then have

$$\begin{aligned} \text{density:} \quad \delta\rho \quad &\to \quad \widetilde{\delta\rho} = 0 \\ \text{pressure:} \quad \delta P \quad &\to \quad \widetilde{\delta P} = \delta P_{\rm nad} \equiv \delta P - c_s^2 \delta\rho \end{aligned} \tag{8}$$

  where $c_s^2 = P'/\rho' =$ adiabatic sound speed.
  - gauge-invariant metric perturbation:

$$\text{curvature:} \quad C \to \zeta \equiv C - \frac{a'}{a}\frac{\delta\rho}{\rho'}$$

  - more generally, for any fluid with density $\rho_\alpha(\tau, \vec{x})$ we can identify the curvature perturbation on uniform-α-density time-slices:

$$\zeta_\alpha \equiv C - \frac{a'}{a}\frac{\delta\rho_\alpha}{\rho'_\alpha}$$

Equating gauge-invariant variables

- Gauge-invariant variables are not unique, and they are not independent
- for example, the curvature on uniform-density time-slices can be written in terms of the longitudinal gauge metric potential and density contrast:

$$\zeta_\alpha \equiv \Phi + \frac{1}{3(1+w_\alpha)}\delta_\alpha$$

for example, for radiation and non-relativistic matter:

$$\zeta_\gamma \equiv \Phi + \frac{1}{4}\delta_\gamma$$
$$\zeta_m \equiv \Phi + \frac{1}{3}\delta_m \qquad (9)$$

multiple component cosmology

- Local energy conservation

$$\rho'_\alpha = -3(1+w_\alpha)(a'/a)\rho_\alpha$$

where $w_\alpha = P_\alpha/\rho_\alpha$ leads to

$$\zeta_\alpha = C + \frac{1}{3(1+w_\alpha)}\frac{\delta\rho_\alpha}{\rho_\alpha}$$

  - curvature perturbation, C, on $\delta\rho_\alpha = 0$ time-slices
  - density perturbation, $\delta\rho_\alpha/\rho_\alpha$, on $C = 0$ time-slices
  - *conserved for barotropic fluids, $P_\alpha(\rho_\alpha)$, on large scales.*

multiple component cosmology

- Primordial plasma (e.g., at epoch of primordial nucleosynthesis when $T \approx 1$ MeV)
  - photons, baryons, neutrinos, cold dark matter + dark energy?
  - total curvature/dimensionless density perturbation:

$$\zeta = \sum_\alpha \frac{1+w_\alpha}{1+w} \zeta_\alpha$$

    conserved on large scales for adiabatic perturbations
  - isocurvature/relative entropy perturbation:

$$S_\alpha = 3\left(\zeta_\alpha - \zeta_\gamma\right)$$

    - for example, matter-isocurvature perturbation:

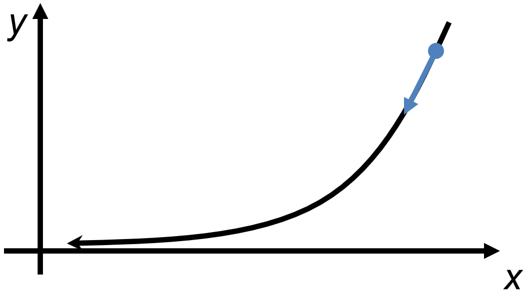
$$S_m = 3\left(\zeta_m - \zeta_\gamma\right) = \frac{\delta\rho_m}{\rho_m} - \frac{3}{4}\frac{\delta\rho_\gamma}{\rho_\gamma}$$

      conserved on large scales

Cosmological perturbations on large scales

- **adiabatic perturbations** *e.g.,* $\delta\left(\frac{n_\gamma}{n_B}\right) \propto \frac{\delta n_\gamma}{n_\gamma} - \frac{\delta n_B}{n_B} = 0$
  - *perturb along the background trajectory*

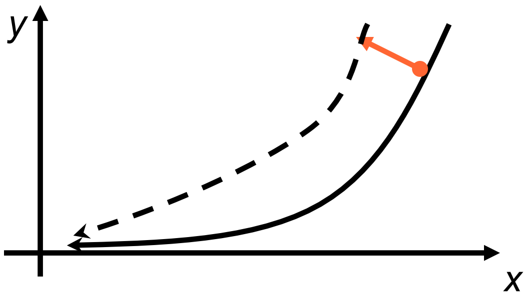
$$\frac{\delta x}{\dot{x}} = \frac{\delta y}{\dot{y}} = \delta T$$



  - *e.g, single-field perturbations along slow-roll attractor*
  - ***adiabatic perturbations stay adiabatic***

- **entropy perturbations**
  - *perturb off the background trajectory*

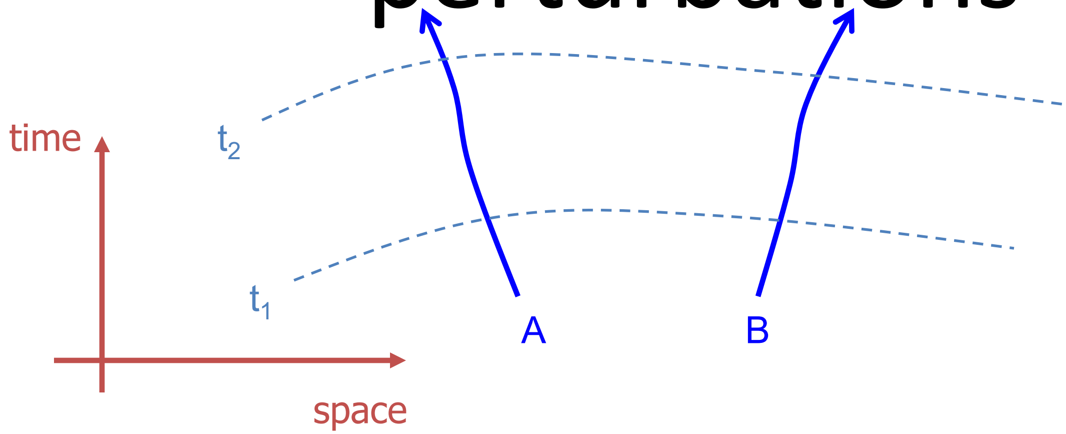
$$\frac{\delta x}{\dot{x}} \neq \frac{\delta y}{\dot{y}}$$



  - *e.g., baryon-photon **isocurvature** perturbation:*

Conserved cosmological perturbations

Lyth & Wands 2003



For every quantity, x, that obeys a **local conservation equation**

$$\frac{dx}{dN} = y(x) \quad , \qquad e.g. \quad \dot{\rho}_m = -3H\rho_m$$

where $dN = Hdt$ is the locally-defined expansion along comoving worldlines

there is a **conserved perturbation** $$\zeta_x \equiv \delta N = \frac{\delta x}{y(x)}$$

where perturbation $\delta x = x_A - x_B$ is a evaluated on hypersurfaces separated by uniform expansion $\Delta N = \Delta \ln a$

examples:

(i) total energy conservation:

$$\frac{d\rho}{dN} = H^{-1}\dot{\rho} = -3(\rho + P)$$

for perfect fluid / adiabatic perturbations, $P=P(\rho)$

$$\Rightarrow \quad \zeta_\rho = \frac{\delta\rho}{3(\rho + P)} \quad \text{conserved}$$

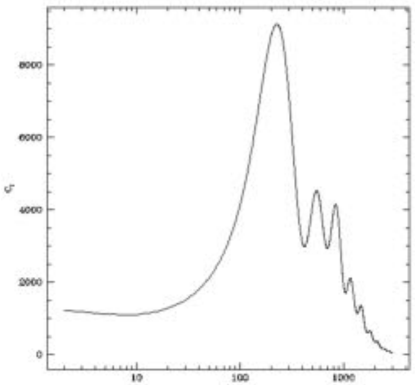
(ii) energy conservation for non-interacting perfect fluids:

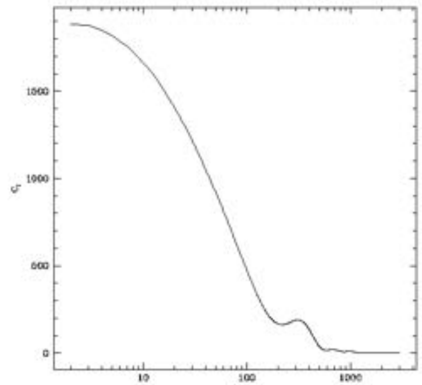
$$H^{-1}\dot{\rho}_i = -3(\rho_i + P_i) \quad \text{where } P_i = P_i(\rho_i) \quad \Rightarrow \quad \zeta_i = \frac{\delta\rho_i}{3(\rho_i + P_i)}$$

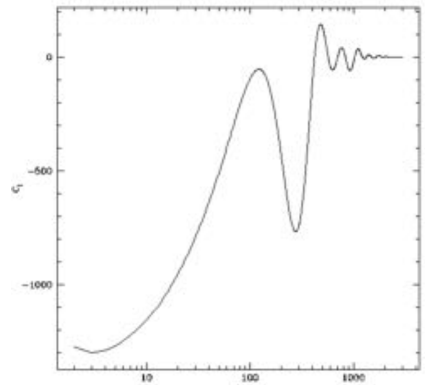
(iii) conserved particle/quantum numbers (e.g., B, B-L,…)

$$H^{-1}\dot{n}_i = -3n_i \quad \Rightarrow \quad \zeta_i = \frac{\delta n_i}{3n_i}$$

microwave background signatures:

$C_l = A^2 \times$ 
adiabatic

$+ B^2 \times$ 
CDM isocurvature

$+ 2\,A\,B\cos\Delta \times$ 
correlation

Bucher, Moodley & Turok '00 } $n_S = 1$

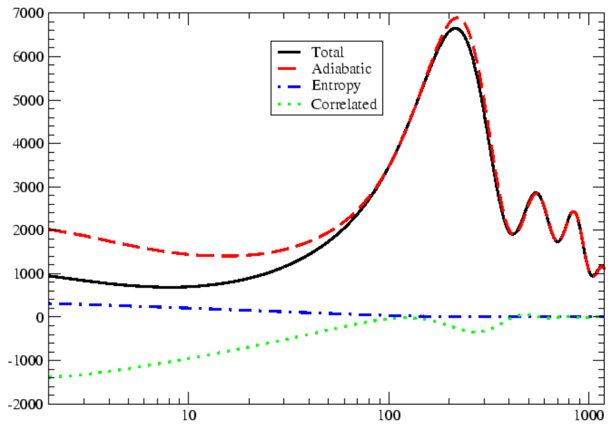
Trotta, Riazuelo & Durrer '01

Amendola, Gordon, Wands & Sasaki '01

best-fit to Boomerang, Maxima & DASI

$B/A = 0.3$, $\cos\Delta = +1$, $n_S = 0.8$

$\omega_b = 0.02$, $\omega_{cdm} = 0.1$, $\Omega_\Lambda = 0.7$



Portsmouth

- island city on the south coast of England
- historic home of the Royal Navy
- University of Portsmouth founded 1992
- Institute of Cosmology and Gravitation established in 2002
- 60+ researchers (academic staff, postdoctoral researchers, phd students and visiting researchers)

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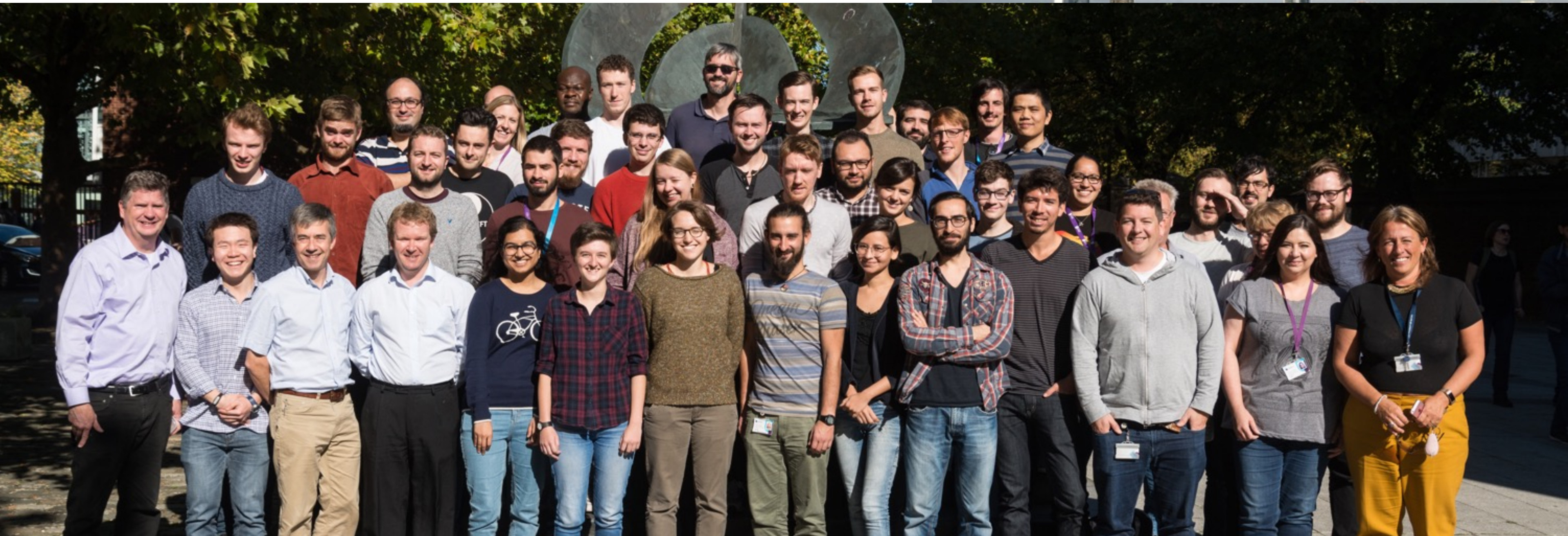

Einstein equations in an arbitrary gauge

Evolution equations

trace and trace-free spatial part of Einstein equations:

$$C'' + 2\mathcal{H}C' - \mathcal{H}A' - (2\mathcal{H}' + \mathcal{H}^2)A = -4\pi G a^2 \left(\delta P + \frac{2}{3}\nabla^2 \Pi\right) ,$$

$$\sigma' + 2\mathcal{H}\sigma - A - C = 8\pi G a^2 \Pi ,$$

Energy+momentum constraints

time-time and time-space components:

$$3\mathcal{H}(C' - \mathcal{H}A) - \nabla^2(C - \mathcal{H}\sigma) = 4\pi G a^2 \delta\rho ,$$

$$C' - \mathcal{H}A = 4\pi G a^2 (\rho + P)(v + B) .$$

Energy+momentum conservation

Fluid continuity and Euler equations:

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) + 3(\rho + P)C' + (\rho + P)\nabla^2(v + E') = 0 ,$$

$$(v + B)' + (1 - 3c_s^2)\mathcal{H}(v + B) + A + \frac{1}{\rho + P}\left(\delta P + \frac{2}{3}\nabla^2 \Pi\right) = 0 .$$

Constraint equations in conformal Newtonian gauge

$$A = \Psi\,, \quad B = 0 \quad , C = \Phi\,, \quad E = 0$$

Energy and momentum constraint equations

$$\begin{aligned} 3\mathcal{H}(\Phi' - \mathcal{H}\Psi) - \nabla^2\Phi &= 4\pi G a^2 \delta\rho\,, \\ \Phi' - \mathcal{H}\Psi &= -4\pi G a^2(\rho + P)V\,. \end{aligned}$$

eliminate $\Phi' - \mathcal{H}\Psi$ gives *Poisson equation*:

$$\frac{\nabla^2}{a^2}\Phi = -4\pi G\delta\rho_c\,,$$

where gauge-invariant *comoving energy density* (i.e., $\delta\rho$ in $v + B = 0$ gauge)

$$\delta\rho_c \equiv \delta\rho + 3\mathcal{H}(\rho + P)V$$

Fluid equations in arbitrary gauge

Fluid continuity and Euler equations:

$$\begin{aligned}\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) + 3(\rho + P)C' + (\rho + P)\nabla^2(v + E') &= 0\,,\\ (v + B)' + (1 - 3c_s^2)\mathcal{H}(v + B) + A + \frac{1}{\rho + P}\left(\delta P + \frac{2}{3}\nabla^2\Pi\right) &= 0\,.\end{aligned}$$

For zero pressure perturbations (e.g., ΛCDM)

$$\begin{aligned}\delta\rho' + 3\mathcal{H}\delta\rho + 3\rho C' + \rho\nabla^2(v + E') &= 0\,,\\ (v + B)' + \mathcal{H}(v + B) + A &= 0\,.\end{aligned}$$

In comoving gauge ($v + B = 0$)

$$\begin{aligned}\delta\rho_c' + 3\mathcal{H}\delta\rho_c + 3\rho C_c' + \rho\nabla^2 V &= 0\,,\\ A_c &= 0\,.\end{aligned}$$

and momentum constraint then reduces to $C_c' = 0$.
In conformal Newtonian gauge ($E = B = 0$) Euler equation becomes

$$V' + \mathcal{H}V + \Psi = 0\,.$$

Recovering Newtonian fluid equations

Changing to density contrast

$$\delta \equiv \frac{\delta\rho}{\rho}$$

we then have for pressureless matter

- Poisson equation for conformal Newtonian potential:

$$\nabla^2\Phi = -\frac{3}{2}\mathcal{H}^2\delta_c$$

- Continuity equation for comoving density:

$$\delta_c' + \nabla^2 V = 0$$

- Euler equation for conformal Newtonian velocity ($\vec{V} = \vec{\nabla} V$):

$$\vec{V}' + \mathcal{H}\vec{V} = -\vec{\nabla}\Psi$$

Coincide exactly to first-order Newtonian perturbation equations in an expanding cosmology, using $\Psi = -\Phi$ for zero anisotropic stress.

Redshift-space distortions

Continuity equation for comoving density contrast:

$$\delta_c' + \nabla^2 V = 0$$

divergence of peculiar velocities, $\theta \equiv \nabla^2 V$, seen in galaxy redshift surveys

$$\langle \theta^2 \rangle = f^2 \langle \delta_c^2 \rangle$$

Important probe of growth of structure given by $f\sigma_8$, where σ_8^2 is the variance of the matter power spectrum on 8 Mpc scales (≈ 1).

Second-order equation for δ

Continuity equation for comoving density contrast:

$$\delta_c' + \nabla^2 V = 0$$

Taking time derivative

$$\delta_c'' + \nabla^2 V' = 0$$

plus Euler equation

$$V' + \mathcal{H} V = -\Psi$$

eliminating V' and V gives

$$\delta_c'' + \mathcal{H}\delta_c' - \nabla^2 \Psi = 0$$

Using $\Psi = -\Phi$ and Poisson equation

$$\nabla^2 \Phi = -\frac{3}{2}\mathcal{H}^2 \delta_c$$

gives linear second-order differential equation

$$\delta_c'' + \mathcal{H}\delta_c' - \frac{3}{2}\mathcal{H}^2 \delta_c = 0$$

Growth of structure

In absence of any pressure perturbations, e.g., ΛCDM

$$\delta_c'' + \mathcal{H}\delta_c' - \frac{3}{2}\mathcal{H}^2\delta_c = 0$$

Growth of structure is independent of scale in ΛCDM cosmology.

$$\delta_k(\tau) = D_+(\tau)\delta_{k,0} + D_-(\tau)\tilde{\delta}_{k,0}$$

- Growing mode $D_+(\tau)$ normalised so that $D_+(\tau_0) = 1$ today
- Decaying mode $D_-(\tau)$ assumed negligible at late times
- In matter-dominated cosmology ($\Omega_m = 1$) then $D_+ \propto a$

$$f \equiv \frac{d \ln D_+}{d \ln a} = 1$$

- In ΛCDM

$$f \simeq \Omega_m^{6/11}$$