Theoretical cosmology Cosmological perturbations

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Theoretical cosmology

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Homogeneous cosmology

Perturbati :heory

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Fourier transforms

Power spectrur

Vector pertu

Tensor perturbat

Metric

perturbation

Geometrical interpretati

uge dependence

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Conformal Newtonian/Longitud

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Einstein equations

Einstein equations in an arbitrary gauge

Recovering Newtonian equations

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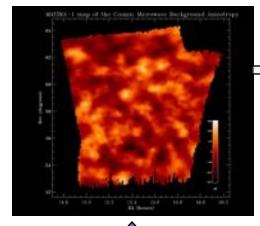
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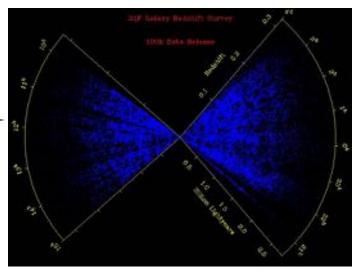
Standard model of structure formation

primordial perturbations

in cosmic microwave background



gravitational _____ instability



large-scale structure of our Universe

new observational data offers precision tests of

- cosmological parameters
- the nature of the primordial perturbations

Inflation:

initial false vacuum state drives accelerated expansion zero-point fluctuations yield spectrum of perturbations

References

- Malik and Wands, Phys Rep 475, 1 (2009), arXiv:0809.4944
- ▶ Bardeen, Phys Rev D22, 1882 (1980)
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- ► Bassett, Tsujikawa and Wands, Rev Mod Phys (2005), astro-ph/0507632

Theoretical cosmology

David Wands

Homogeneous cosmology

Perturbation heory

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Scalar perti

Fourier transfor

Vector perturb

vector perturb

Metric

erturbations

eometrical interpretation

auge dependence

Conformal

Vewtonian/Longitue gauge

> form-density gau ting gauge-invari

Einstein equations

Einstein equations in an

Recovering Newtonian fl

Redshift-space di



Outline

Theoretical cosmology

David Wands

Homogeneous cosmology

Perturbation theory

Metric perturbations

Einstein equations

FLRW metric

- ▶ 4D spacetime split into 1+3
- Friedmann-Lemaitre-Robertson-Walker (FLRW) line element:

$$ds^2 = -c^2 dt^2 + a^2(t) dX^2$$

- ▶ time + homogeneous and isotropic space
- dynamical scale factor, a(t), where $a_0 = 1$ today
- ► maximally-symmetric 3-space, curvature *K*

$$dX^{2} = \frac{dr^{2}}{1 - Kr^{2}} + r^{2} \left(d\theta^{2} + \sin^{2}\theta d\phi^{2}\right)$$

Theoretical cosmology

David Wands

Homogeneous cosmology

erturbatio

heory

Fourier transforms
Power spectrum

Vector perturbations Tensor perturbations

Metric

erturbations

auge dependence

articular gauges

onformal ewtonian/Longitud auge

Iniform-density gau quating gauge-invar riables

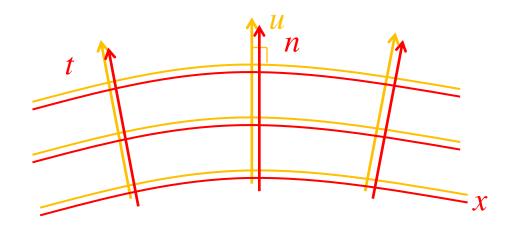
Einstein equations

Einstein equations in an arbitrary gauge Recovering Newtonian fluid

Redshift-space distort

Recovering Newtoni Iuid equations





FRW cosmology preferred coordinates for homogeneous and isotropic space

preferred space+time split in FRW cosmology breaks symmetry of Einstein's theory

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$$dX^2 = \frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right)$$

▶ alternative (conformal) time coordinate, $d\tau = c dt/a$:

$$ds^2 = a^2(\tau) \left[-d\tau^2 + dX^2 \right]$$

Theoretical cosmology

David Wands

Homogeneous cosmology

erturbation

theory

Power spectrum

ector perturbations

ensor perturbations

Metric

erturbations

Geometrical interpretatior Gauge dependence

articular gauges Conformal

lewtonian/Longit auge

Jniform-density gau quating gauge-invari griables

Einstein equations

Einstein equations in an arbitrary gauge Recovering Newtonian fluid

> Redshift-space distortion Recovering Newtonian



FLRW metric

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▶ alternative (conformal) time coordinate, $d\tau = c dt/a$:

$$ds^2 = a^2(\tau) \left[-d\tau^2 + dX^2 \right]$$

ightharpoonup Henceforth assume K=0, flat space

Theoretical cosmology

David Wands

Homogeneous cosmology

erturbation

theory

Fourier transforms
Power spectrum

Vector perturbation

Metric

erturbations

Gauge dependence

Particular gauges Conformal

Newtonian/Long gauge

Jniform-density ga quating gauge-inv giables

instein equations

Einstein equations

Einstein equations in an arbitrary gauge Recovering Newtonian fluid equations

> Recovering Newtonian fluid equations



Scalar perturbations

- Scalar quantity, e.g., density at fixed point P is invariant under change of coordinates
- Split into background (homogeneous) part and a perturbation (inhomogeneous):

$$\rho(t, \vec{x}) = \bar{\rho}(t) + \delta \rho(t, \vec{x})$$

expand perturbation order-by-order in a small parameter. ε :

$$\delta \rho(t, \vec{x}) = \varepsilon \delta_1 \rho(t, \vec{x}) + \frac{1}{2} \varepsilon^2 \delta_2 \rho(t, \vec{x}) + \dots$$

 \blacktriangleright keep only terms at first order in $\varepsilon \Rightarrow$ linear peturbations

$$\delta \rho(t, \vec{x}) = \varepsilon \delta_1 \rho(t, \vec{x})$$

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Expanding equations order-by-order

e.g., non-relativistic continuity equation for density $\rho(t,\vec{x})$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{1}$$

expand density and velocity order-by-order

$$\rho(t, \vec{x}) = \bar{\rho}(t) + \varepsilon \delta_1 \rho(t, \vec{x}) + \frac{1}{2} \varepsilon^2 \delta_2 \rho(t, \vec{x}) + \dots$$
$$\vec{v}(t, \vec{x}) = \varepsilon \delta_1 \vec{v}(t, \vec{x}) + \frac{1}{2} \varepsilon^2 \delta_2 \vec{v}(t, \vec{x}) + \dots$$

substitute into Eq. (1)

$$\frac{\partial}{\partial t} \left(\bar{\rho} + \varepsilon \delta_1 \rho + \frac{1}{2} \varepsilon^2 \delta_2 \rho + \dots \right)$$

$$+ \vec{\nabla} \cdot \left[\left(\bar{\rho} + \varepsilon \delta_1 \rho + \frac{1}{2} \varepsilon^2 \delta_2 \rho + \dots \right) \right.$$

$$\times \left(\varepsilon \delta_1 \vec{v} + \frac{1}{2} \varepsilon^2 \delta_2 \vec{v} + \dots \right) \right] = 0$$

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Perturbation equations order-by-order

ightharpoonup collect terms order-by-order in ε

$$\frac{\partial}{\partial t} \bar{\rho}
+\varepsilon \left\{ \frac{\partial}{\partial t} \delta_1 \rho + \vec{\nabla} \cdot (\bar{\rho} \delta_1 \vec{v}) \right\}
+ \frac{1}{2} \varepsilon^2 \left\{ \frac{\partial}{\partial t} \delta_2 \rho + \vec{\nabla} \cdot (\bar{\rho} \delta_2 \vec{v} + 2 \delta_1 \rho \delta_1 \vec{v}) \right\} + \dots = 0$$

> solve order-by-order in ε

$$\frac{\partial}{\partial t} \vec{\rho} = 0 \Rightarrow \vec{\rho} = 0$$

$$\frac{\partial}{\partial t} \delta_1 \rho + C \vec{\nabla} \cdot \delta_1 \vec{v} = 0$$

$$\frac{\partial}{\partial t} \delta_2 \rho + C \vec{\nabla} \cdot \delta_2 \vec{v} = -2C \vec{\nabla} \cdot (\delta_1 \rho \, \delta_1 \vec{v})$$

Theoretical cosmology

David Wands

Homogeneous cosmology

theory

Scalar perturbations

Power spectrum

Tensor perturbations

Metric

conturbations

Geometrical interpretation Gauge dependence

articular gauges Conformal

Newtonian/Longitudin gauge Uniform-density gauge

Uniform-density gauge Equating gauge-invarian variables

Einstein equations

Einstein equations in an arbitrary gauge Recovering Newtonian fluid

Redshift-space

Recovering Newtonian



Fourier transform

► Field in real space is an integral over Fourier modes:

$$\delta \rho(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \, \delta \rho_{\vec{k}}(t) \, e^{i\vec{k}.\vec{x}}$$

► Fourier modes are eigenfunctions of the spatial Laplacian:

$$\nabla^2 \left(e^{i\vec{k}.\vec{x}} \right) = -k^2 e^{i\vec{k}.\vec{x}}$$

which provide a complete orthonormal basis:

$$\int d^3x \, e^{i\vec{k}_1 \cdot \vec{x}} \, e^{i\vec{k}_2 \cdot \vec{x}} = (2\pi)^3 \delta^{(3)} \left(\vec{k}_1 - \vec{k}_2 \right)$$

► Coefficient in Fourier space is integral over real space:

$$\delta
ho_{\vec{k}}(t) = \int d^3x \, \delta
ho(t, \vec{x}) \, e^{-i \vec{k}.\vec{x}}$$

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David Wands

Homogeneous cosmology

Perturbation heory

Scalar perturbation

Fourier transforms

Vector perturbations
Tensor perturbations

Vetric

perturbations
Commercial interpre

Gauge dependence

Conformal Newtonian/Long

> auge Jniform-density gauge quating gauge-invarian

Einstein equations

Einstein equations in an arbitrary gauge Recovering Newtonian fluid

Redshift-space dist

Recovering Newtonian luid equations



Statistical distribution

- theory describes properties of distribution = ensemble, assumed isotropic (⟨...⟩ = average over all possible realisations)
- observations describe one realisation from the distribution

Theoretical cosmology

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Homogeneous cosmology

theory

Scalar perturbations

Fourier transforms

Vector perturbations

1etric

perturbations

Geometrical interpretation

auge dependence

articular gauges Conformal

onformal lewtonian/Longitudi auge

Uniform-density ga

instein equation

Einstein equations in an arbitrary gauge Recovering Newtonian fluid

Redshift-space

ecovering Newtoni uid equations



Power spectrum

defined by the correlation of two modes in Fourier space:

$$\langle \delta \rho_{\vec{k}_1} \delta \rho_{\vec{k}_2} \rangle = (2\pi)^3 P_{\rho}(k_1) \delta^3 \left(\vec{k}_1 + \vec{k}_2 \right)$$

note: $P_{\rho}(k)$ only a function of wavenumber k, not wavevector \vec{k} , for an isotropic distribution

▶ Variance in real space: (exercise for reader!)

$$\begin{split} \langle \delta \rho^{2}(\vec{x}) \rangle &= \langle \int \frac{d^{3}\vec{k}_{1}d^{3}\vec{k}_{2}}{(2\pi)^{6}} \delta \rho_{\vec{k}_{1}} \delta \rho_{\vec{k}_{2}} e^{i(\vec{k}_{1} + \vec{k}_{2}) \cdot \vec{x}} \rangle \\ &= \int \frac{d^{3}\vec{k}_{1}d^{3}\vec{k}_{2}}{(2\pi)^{6}} \langle \delta \rho_{\vec{k}_{1}} \delta \rho_{\vec{k}_{2}} \rangle e^{i(\vec{k}_{1} + \vec{k}_{2}) \cdot \vec{x}} \\ &= \int \frac{d^{3}\vec{k}_{1}}{(2\pi)^{3}} P_{\rho}(k_{1}) = \int d \ln k_{1} \mathcal{P}_{\rho}(k_{1}) \end{split}$$

ightharpoonup dimensionless power spectrum per log k:

$$\mathcal{P}_{
ho}(k) = rac{4\pi k^3}{(2\pi)^3} P_{
ho}(k)$$

Theoretical cosmology

David Wands

Homogeneous cosmology

heory

Scalar perturbations

Power spectrum

ector perturbations
ensor perturbations

Metric

perturbation

Gauge dependence Particular gauges

Conformal Newtonian/Lor gauge

> Uniform-density gau Equating gauge-invar

Einstein equations

Einstein equations in an arbitrary gauge Recovering Newtonian fluid

Redshift-space distortions
Recovering Newtonian

Higher-order statistics

▶ Bispectrum

$$\langle \delta \rho_{\vec{k}_1} \delta \rho_{\vec{k}_2} \delta \rho_{\vec{k}_3} \rangle = (2\pi)^3 B_{\rho}(k_1, k_{2,3}) \delta^3 \left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 \right)$$

Bispectrum is zero for *Gaussian* perturbations (and for all odd moments)

▶ We will take *first-order* perturbations to be *Gaussian*:

$$\langle \delta_1 \rho_{\vec{k}_1} \delta_1 \rho_{\vec{k}_2} \delta_1 \rho_{\vec{k}_3} \rangle = 0$$

Second- and higher-order perturbations are non-Gaussian.

$$\langle \delta_2 \rho_{\vec{k}_1} \delta_1 \rho_{\vec{k}_2} \delta_1 \rho_{\vec{k}_3} \rangle \neq 0$$

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David Wands

Homogeneous cosmology

rturbatio eorv

Scalar perturbations

Power spectrum

Vector perturbations

/letric

perturbation

Geometrical interpretation

Sauge dependenci Particular gauges

Conformal

Newtonian/Longi gauge

Uniform-density ga

riables

Einstein equations

Einstein equations in an arbitrary gauge Recovering Newtonian fluid

Redshift-space distorated Recovering Newtoni

Vector perturbations

- decompose any 3-vector: $\vec{V} = \vec{\nabla} V^{(s)} + \vec{V}^{(v)}$
 - scalar (longitudinal/potential) flow: $\vec{\nabla} \times \vec{\nabla} V^{(s)} = 0$
 - ightharpoonup vector (transverse/divergence-free) flow: $\vec{
 abla}\cdot\vec{V}^{(
 u)}=0$
- ► Fourier transform
 - scalar

$$V^{(s)}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} V_{\vec{k}}^{(s)}(t) e^{i\vec{k}.\vec{x}}$$

vector

$$ec{V}^{(v)}(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left\{ V_{\vec{k}}^{(v)}(t) \vec{e}_{\vec{k}} + \tilde{V}_{\vec{k}}^{(v)}(t) \vec{e}_{\vec{k}} \right\} e^{i\vec{k}.\vec{x}}$$

where $\vec{e}_{\vec{k}}$ and $\vec{\tilde{e}}_{\vec{k}}$ are orthonormal polarisation vectors:

$$\vec{e}_{\vec{k}} \cdot \vec{e}_{\vec{k}} = \vec{\tilde{e}}_{\vec{k}} \cdot \vec{\tilde{e}}_{\vec{k}} = 1, \quad \vec{e}_{\vec{k}} \cdot \vec{\tilde{e}}_{\vec{k}} = 0$$

transverse to wavevector \vec{k} :

$$\vec{k} \cdot \vec{e}_{\vec{k}} = \vec{k} \cdot \vec{\tilde{e}}_{\vec{k}} = 0$$

Theoretical cosmology

David Wands

Homogeneous cosmology

Perturbat heory

Scalar perturbations

Power spectrum

Vector perturbations

Tensor perturbations

Metric

perturbatio

Gauge dependence

articular gauges Conformal

Newtonian/Longitudir gauge

Iniform-density gaug quating gauge-invaria riables

Einstein equations

Einstein equations in an arbitrary gauge Recovering Newtonian fluid

Redshift-space distortio



Vector perturbations

putting it all together

$$\vec{V}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left\{ i\vec{k} \, V_{\vec{k}}^{(s)}(t) + \vec{e}_{\vec{k}} \, V_{\vec{k}}^{(v)}(t) + \vec{\tilde{e}}_{\vec{k}} \, \tilde{V}_{\vec{k}}^{(v)}(t) \right\} e^{i\vec{k}.\vec{x}}$$

Theoretical cosmology

David Wands

Homogeneous cosmology

Perturbation theory

Scalar perturbation

Fourier transforms

Power spectrum

Vector perturbations

ensor perturbations

Metric

perturbations

Geometrical interpretati

Sauge dependence

Particular gauges

Conformal

Newtonian/Longitudina gauge

iform-density gaug ating gauge-invaria

nstein equation

Einstein equations in an arbitrary gauge

equations equations

Redshift-space dist



Tensor perturbations

decompose any 3-tensor:

$$T_{ij} = \delta_{ij}C + \nabla_i\nabla_jS + (1/2)(\nabla_iV_j + \nabla_jV_i) + h_{ij}$$

- ► scalars C and S are longitudinal/potential
- vector V_i is transverse: $\nabla^i V_i = 0$
- tensor hii is transverse and trace-free:

$$\nabla^i h_{ij} = \nabla^j h_{ij} = 0 \,, \qquad h^i_i = 0 \label{eq:poisson}$$

Theoretical cosmology

David Wands

Homogeneous cosmology

Perturbation Theory

Scalar perturbations

Power spectrum

Tensor perturbations

lotric

perturbation

Geometrical interpreta

uge dependence

ticular gauges

onformal lewtonian/Longitudi auge

niform-density gau uating gauge-invari

Einstein equations

Einstein equations in an arbitrary gauge

Recovering Newtonia equations

Redshift-space distortion



Tensor perturbations

► Fourier transform:

$$h_{ij}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \left\{ h_{\vec{k}}^{(+)}(t) q_{\vec{k}\,ij}^{(+)} + h_{\vec{k}}^{(\times)}(t) q_{\vec{k}\,ij}^{(\times)} \right\} e^{i\vec{k}.\vec{x}}$$

where polarisation tensors

$$q_{\vec{k}\,ij}^{(+)} = \frac{1}{\sqrt{2}} \left(e_{\vec{k}\,i} e_{\vec{k}\,j} - \tilde{e}_{\vec{k}\,i} \tilde{e}_{\vec{k}\,j} \right) q_{\vec{k}\,ij}^{(\times)} = \frac{1}{\sqrt{2}} \left(e_{\vec{k}\,i} \tilde{e}_{\vec{k}\,j} + \tilde{e}_{\vec{k}\,i} e_{\vec{k}\,j} \right)$$

and $e_{\vec{k}\,i}$ and $\tilde{e}_{\vec{k}\,i}$ are orthonormal, transverse vectors, such that (exercise for reader!)

$$q_{\vec{k}}^{(+)\,ij}q_{\vec{k}\,ij}^{(+)} = q_{\vec{k}}^{(\times)\,ij}q_{\vec{k}\,ij}^{(\times)} = 1\,, \quad q_{\vec{k}}^{(+)\,ij}q_{\vec{k}\,ij}^{(\times)} = 0$$

tracefree $q_{\vec{k},i}^{(+)i} = q_{\vec{k},i}^{(\times)i} = 0$ and transverse to \vec{k} :

$$k^{i}q_{\vec{k}\,ij}^{(+)}=k^{i}q_{\vec{k}\,ij}^{(imes)}=0$$

Theoretical cosmology

David Wands

Homogeneous cosmology

erturbatio neorv

Scalar perturbations

Power spectrum

Tensor perturbations

Vietric

perturbations
Geometrical interpreta

Gauge dependence Particular gauges

Conformal Newtonian/Longitu gauge

Uniform-density gauge Equating gauge-invariant

Einstein equations

Einstein equations in an arbitrary gauge Recovering Newtonian fluid equations

> Redshift-space distortion Recovering Newtonian



Metric perturbations

Split metric into spatially-flat FLRW background and inhomogeneous perturbation:

$$g_{\mu\nu}=ar{g}_{\mu
u}+\delta g_{\mu
u}$$
 .

Background:

$$\bar{g}_{00} = a^2, \quad \bar{g}_{0i} = 0, \quad \bar{g}_{ij} = a^2 \delta_{ij}$$
 (2)

Perturbation:

$$\begin{array}{lcl} \delta g_{00} & = & 2a^2A \\ \delta g_{0i} & = & a^2 \left(\nabla_i B - S_i\right) \\ \delta g_{ij} & = & a^2 \left(2C\delta_{ij} + 2\nabla_i \nabla_j E + \nabla_i F_j + \nabla_j F_i + h_{ij}\right) \end{array}$$

- ▶ 4 scalars: *A*, *B*, *C*, *E*
- \triangleright 2 vectors: S_i, F_i
- ▶ 1 tensor: *h*_{ii}

Theoretical cosmology

David Wands

Homogeneous cosmology

erturbat eorv

Scalar porturba

Fourier transforms
Power spectrum

ector perturbations

Metric perturbations

metrical interpretation

articular gauges

Conformal Newtonian/Long

auge Iniform-density gauge

quating gauge-inva ariables

Einstein equations

Einstein equations in an arbitrary gauge Recovering Newtonian fluid equations

> Redshift-space distortion Recovering Newtonian

fluid equations



Metric perturbations

Perturbed line-element including only scalar perturbations:

$$ds^{2} = g_{\mu\nu}dx^{\mu}dx^{\nu}$$

$$= a^{2}(\tau)\left\{-(1+2A)d\tau^{2} + 2(\partial_{i}B)dx^{i}d\tau + \left[(1+2C)\delta_{ij} + 2(\partial_{ij}E)\right]\right\}dx^{i}dx^{j}$$

where four scalar perturbations are

- ightharpoonup A = lapse perturbation
- $\triangleright \partial_i B = \partial B/\partial x^i = \text{shift perturbation}$
- ightharpoonup C = spatial curvature perturbation
- $ightharpoonup \partial_{ij}E = \partial^2 E/\partial x^i \partial x^j = \text{off-diagonal spatial perturbation}$

Theoretical cosmology

David Wands

Homogeneous cosmology

Perturbat heory

Scalar perturbations

ower spectrum

Vector perturbations Tensor perturbations

Metric perturbations

eometrical interpretation

Particular gauge

articular gauge Conformal

lewtonian/Longitudi auge Iniform donsity gaug

Jniform-density gauge quating gauge-invarian

Einstein equations

Einstein equations in an arbitrary gauge

Recovering Newtoni equations

dshift-space distortio



Geometrical interpretation

► Temporal gauge (time-slicing) in 4D spacetime defines a hypersurface orthogonal 4-vector field:

$$N_{\mu} \propto rac{\partial au}{\partial x^{\mu}}$$

normalise such that $N_{\mu}N^{\mu}=-1$.

ightharpoonup intrinsic curvature of constant au hypersurfaces:

$$^{(3)}R = -\frac{4}{a^2}\nabla^2 C$$

ightharpoonup expansion of constant au hypersurfaces:

$$\theta = \frac{3}{a} \left(\frac{a'}{a} (1 - A) + C' + \frac{1}{3} \nabla^2 \sigma \right)$$

shear:

$$\sigma_{ij} = \left(\nabla_i \nabla_j - \frac{1}{3} \nabla^2\right) \sigma \,, \qquad \sigma = E' - B$$

acceleration:

$$a_i = \nabla_i A$$

Theoretical cosmology

David Wands

Homogeneous cosmology

'erturbatioı heory

Scalar perturbations

Fourier transform Power spectrum

Vector perturbations

etric sturbatio

Geometrical interpretation

Gauge dependence

articular gauges

Conformal

wtonian/Longitudii uge

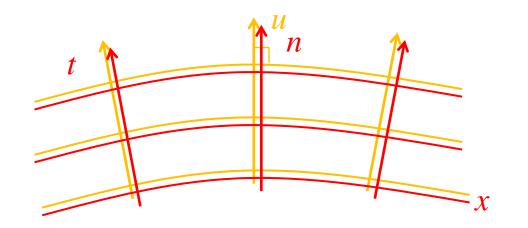
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Einstein equations

Einstein equations in an arbitrary gauge Recovering Newtonian fluid

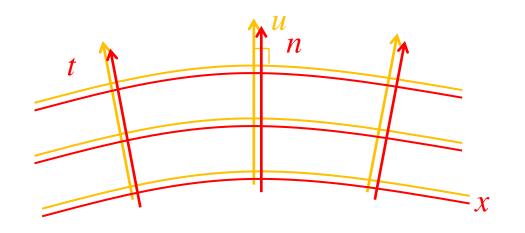
Redshift-space distortion

Recovering Newtonian fluid equations



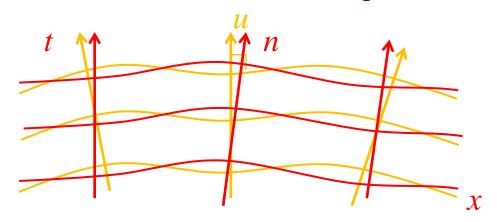
FRW cosmology preferred coordinates for homogeneous and isotropic space

preferred space+time split in FRW cosmology breaks symmetry of Einstein's theory



FRW cosmology

no unique choice of time (slicing) and space coordinates (threading) in an inhomogeneous universe



FRW cosmology + perturbations

arbitrary gauge (t,x)

gauge problem: find different perturbations in different gauges

Gauge dependence

 \triangleright Scalar quantity, e.g., density, $\rho|_P$, at given point P is invariant

$$\rho(\tau, \vec{x})|_P = \tilde{\rho}(\tilde{\tau}, \tilde{\vec{x}})|_P$$

under first-order (scalar) change of coordinates:

$$\tilde{\vec{x}} = \tau + \delta \tau(\tau, \vec{x})$$

$$\tilde{\vec{x}} = \vec{x} + \vec{\nabla} \delta x(\tau, \vec{x})$$

but background-perturbation split is gauge-dependent

$$\rho_{0}(\tau) + \delta \rho|_{P} = \rho_{0}(\tilde{\tau}) + \widetilde{\delta \rho}|_{P}
\Rightarrow \widetilde{\delta \rho}|_{P} = \delta \rho|_{P} + \rho_{0}(\tau) - \rho_{0}(\tilde{\tau})
= \delta \rho|_{P} - \rho'_{0}\delta \tau$$
(3)

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Gauge dependence

Linear gauge transformation rules

Coordinate change:

time-slicing:
$$\tilde{\tau} \rightarrow \tau + \delta \tau(\tau, \vec{x})$$
 spatial-threading: $\tilde{\vec{x}} \rightarrow \vec{x} + \vec{\nabla} \delta x(\tau, \vec{x})$

Gauge transformations:

density:
$$\widetilde{\delta\rho} = \delta\rho - \rho'\delta\tau$$

pressure: $\widetilde{\delta P} = \delta P - P'\delta\tau$
velocity: $\widetilde{v}^i = v^i + \partial^i\delta x$ (4)

including three metric transformations independent of spatial-threading:

lapse:
$$\tilde{A} = A - \frac{a'}{a}\delta\tau - \delta\tau'$$

curvature: $\tilde{C} = C - \frac{a'}{a}\delta\tau$

shear: $\tilde{\sigma} = \tilde{E}' - \tilde{B} = \sigma - \delta\tau$ (5)

Theoretical cosmology

David Wands

Homogeneous cosmology

Perturbation heory

theory

Scalar perturbation

Power spectrum ector perturbations

Metric

Geometrical interpretation

Gauge dependence

articular gauges

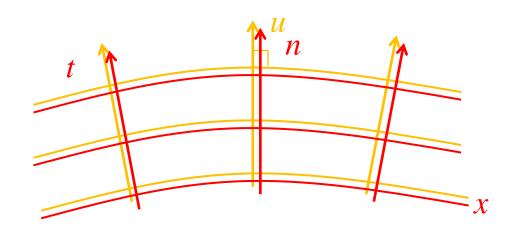
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iform-density gau ating gauge-invar

Einstein equations

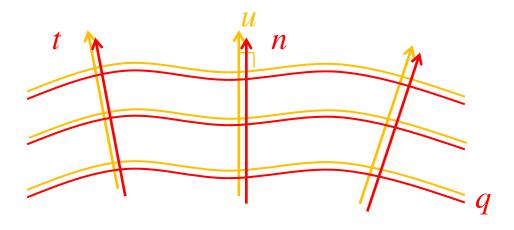
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> Recovering Newtonian fluid equations



FRW cosmology

synchronous+comoving with pressureless cold dark matter time-slicing orthogonal to comoving worldlines



FRW cosmology + perturbations

comoving-Lagrangian coordinates (*t*,*q*)

Conformal Newtonian/Longitudinal gauge

- pick a gauge to completely fix the coordinates
- ▶ for example: *longitudinal gauge (zero-shear time-slices)*:
 - ightharpoonup set $\sigma
 ightharpoonup \tilde{\sigma} = 0$ which requires a transform $\delta \tau = \sigma$
 - we then have

density:
$$\delta \equiv \frac{\delta \rho}{\rho} \rightarrow \widetilde{\delta} = \delta - \frac{\rho'}{\rho} \sigma$$
 (6)

including two gauge-invariant metric perturbations:

lapse:
$$A \rightarrow \Psi \equiv A - \frac{a'}{a}\sigma - \sigma'$$

curvature: $C \rightarrow \Phi \equiv C - \frac{a'}{a}\sigma$ (7

Theoretical cosmology

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Scalar perturbations

Ower spectrum

ector perturbations

Metric

erturbations

Geometrical interpretation Gauge dependence

Conformal

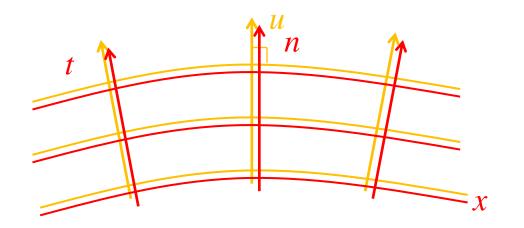
Newtonian/Longitudinal gauge

Uniform-density gauge Equating gauge-invariant variables

Einstein equations

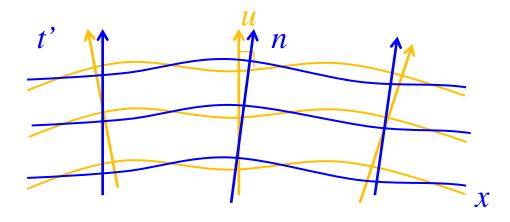
Einstein equations in an arbitrary gauge Recovering Newtonian fluid

Redshift-space distortions



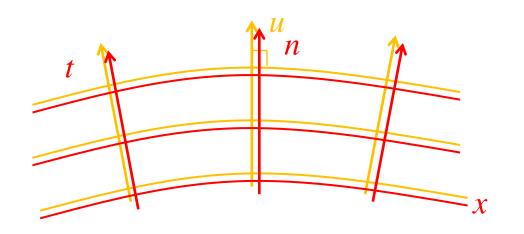
FRW cosmology

Poisson = conformal Newtonian = longitudinal gauge hypersurface-orthogonal 4-vector field n is shear-free



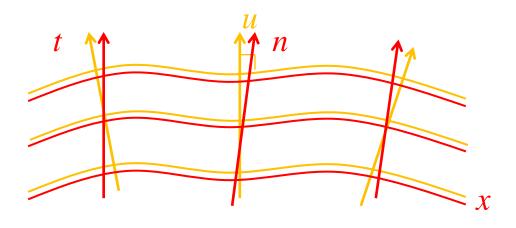
FRW cosmology + perturbations

Poisson gauge coordinates (t',x)



FRW cosmology

time-slicing orthogonal to comoving worldlines spatial threading is same as Poisson gauge (Eulerian, not Lagrangian)



FRW cosmology + perturbations

total-matter coordinates (t,x)

Standard Newtonian+Gaussian initial fields

Gaussian primordial metric fluctuations $\zeta(x)$ from inflation + linear Einstein-Boltzmann code (e.g., CMBfast, CAMB, CLASS)

Gaussian initial Newtonian potential

$$\Phi = (3/5)\zeta$$



$$\nabla^2 \Phi = 4\pi G a^2 \bar{\rho} \delta$$



Gaussian initial displacement

$$\vec{\nabla} \cdot \vec{\Psi} = -\delta$$

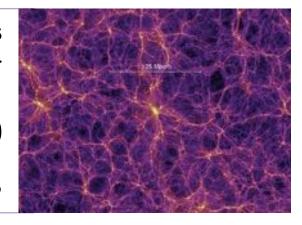


Newtonian N-body simulations

$$\nabla^2 \Phi = 4\pi G a^2 \bar{\rho} \delta$$

$$\dot{\delta} + \vec{\nabla} \cdot ((1+\delta))\vec{v} = 0$$

$$|\dot{ec{v}}+\mathcal{H}ec{v}+(ec{v}.ec{
abla})ec{v}=-ec{
abla}\Phi$$



Uniform-density gauge

- pick a gauge to completely fix the coordinates
- ► for example: *uniform-density time-slices*:
 - set $\delta
 ho o \delta
 ho = 0$ which requires a transform $\delta au = \delta
 ho /
 ho'$
 - we then have

density:
$$\delta \rho \rightarrow \widetilde{\delta \rho} = 0$$

pressure: $\delta P \rightarrow \widetilde{\delta P} = \delta P_{\rm nad} \equiv \delta P - c_s^2 \delta \rho$ (8)

where $c_s^2 = P'/\rho' = \text{adiabatic sound speed}$.

▶ gauge-invariant metric perturbation:

curvature:
$$C \rightarrow \zeta \equiv C - \frac{a'}{a} \frac{\delta \rho}{\rho'}$$

more generally, for any fluid with density $\rho_{\alpha}(\tau, \vec{x})$ we can identify the curvature perturbation on uniform- α -density time-slices:

$$\zeta_{\alpha} \equiv C - \frac{\mathsf{a}'}{\mathsf{a}} \frac{\delta \rho_{\alpha}}{\rho_{\alpha}'}$$

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Homogeneous cosmology

Perturbation heory

Scalar perturbations

Fourier transform: Power spectrum

Vector perturbation

Metric

perturbations

Geometrical interpretation Gauge dependence

Particular gauges

Conformal

Newtonian/Longitudina gauge

Uniform-density gauge

variables

Einstein equatio

Einstein equations in an arbitrary gauge Recovering Newtonian fluid

Redshift-space distorti

Recovering Newtonian



Equating gauge-invariant variables

- Gauge-invariant variables are not unique, and they are not independent
- ► for example, the curvature on uniform-density time-slices can be written in terms of the longitudinal gauge metric potential and density contrast:

$$\zeta_{lpha} \equiv \Phi + rac{1}{3(1+w_{lpha})}\delta_{lpha}$$

for example, for radiation and non-relativistic matter:

$$\zeta_{\gamma} \equiv \Phi + \frac{1}{4} \delta_{\gamma}$$

$$\zeta_{m} \equiv \Phi + \frac{1}{3} \delta_{m} \tag{9}$$

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Homogeneous cosmology

erturbati neorv

Scalar perturbation

Vector perturbat

etric

perturbation

Geometrical interpretation

Particular gauges

Conformal Newtonian/Longitudi gauge

Uniform-density gauge Equating gauge-invariant

variables

instein equation

Einstein equations in an arbitrary gauge
Recovering Newtonian fluid

edshift-space distortion ecovering Newtonian

uid equations



multiple component cosmology

Local energy conservation

$$\rho_{\alpha}' = -3(1+w_{\alpha})(a'/a)\rho_{\alpha}$$

where $w_{\alpha} = P_{\alpha}/\rho_{\alpha}$ leads to

$$\zeta_{\alpha} = C + \frac{1}{3(1+w_{\alpha})} \frac{\delta \rho_{\alpha}}{\rho_{\alpha}}$$

- lacktriangle curvature perturbation, C, on $\delta
 ho_{lpha} = 0$ time-slices
- density perturbation, $\delta \rho_{\alpha}/\rho_{\alpha}$, on C=0 time-slices
- conserved for barotropic fluids, $P_{\alpha}(\rho_{\alpha})$, on large scales.

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Perturbation heorv

Scalar perturbations

Fourier transforms

Vector perturbatio

Tensor perturbat

Metric

perturbations

Geometrical interpretation

Particular gauges

Conformal Newtonian/Long gauge

Jniform-density gauge

Equating gauge-invariant

Einstein equations

Einstein equations in an

arbitrary gauge Recovering Newtonian flu

quations Redshift-space dist

covering Newtonian



multiple component cosmology

- Primordial plasma (e.g., at epoch of primordial nucleosynthesis when $T \approx 1 \text{ MeV}$)
 - photons, baryons, neutrinos, cold dark matter + dark energy?
 - total curvature/dimensionless density perturbation:

$$\zeta = \sum_{\alpha} \frac{1 + w_{\alpha}}{1 + w} \zeta_{\alpha}$$

conserved on large scales for adiabatic perturbations

isocurvature/relative entropy perturbation:

$$S_{\alpha} = 3\left(\zeta_{\alpha} - \zeta_{\gamma}\right)$$

▶ for example, matter-isocurvature perturbation:

$$S_m = 3(\zeta_m - \zeta_\gamma) = \frac{\delta \rho_m}{\rho_m} - \frac{3}{4} \frac{\delta \rho_\gamma}{\rho_\gamma}$$

conserved on large scales

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Homogeneous cosmology

erturbations

Carlan a antonibas

Scalar perturbations
Fourier transforms

Power spectrum ector perturbations

Metric

perturbations

Geometrical interpretation

Particular gauges Conformal

Newtonian/Longito gauge

Uniform-density gauge Equating gauge-invariant

variables

Einstein equations

Einstein equations in an arbitrary gauge Recovering Newtonian fluid equations

> Redshift-space distorti Recovering Newtonian



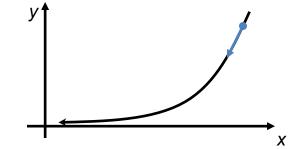
Cosmological perturbations on large scales

adiabatic perturbations

e.g.,
$$\delta \left(\frac{n_{\gamma}}{n_{B}}\right) \propto \frac{\delta n_{\gamma}}{n_{\gamma}} - \frac{\delta n_{B}}{n_{B}} = 0$$

perturb along the background trajectory

$$\frac{\delta x}{\dot{x}} = \frac{\delta y}{\dot{y}} = \delta T$$



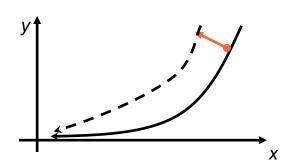
- e.g, single-field perturbations along slow-roll attractor
- adiabatic perturbations stay adiabatic

entropy perturbations

perturb off the background trajectory

$$\frac{\delta x}{\dot{x}} \neq \frac{\delta y}{\dot{y}}$$

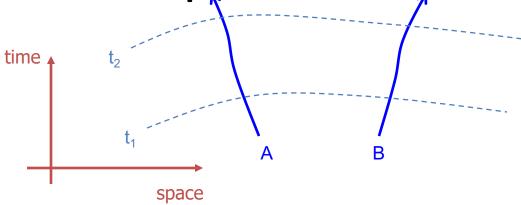
e.g., baryon-photon isocurvature perturbation:



Conserved cosmological

perturbations

Lyth & Wands 2003



For every quantity, x, that obeys a **local conservation equation**

$$\frac{dx}{dN} = y(x)$$
 , e.g. $\dot{\rho}_m = -3H\rho_m$

where dN = Hdt is the locally-defined expansion along comoving worldlines

there is a conserved perturbation

$$\zeta_x \equiv \delta N = \frac{\delta x}{y(x)}$$

where perturbation $\delta x = x_A - x_B$ is a evaluated on hypersurfaces separated by uniform expansion $\Delta N = \Delta \ln a$

examples:

(i) total energy conservation:

$$\frac{d\rho}{dN} = H^{-1}\dot{\rho} = -3(\rho + P)$$

for perfect fluid / adiabatic perturbations, $P=P(\rho)$

$$\Rightarrow \zeta_{\rho} = \frac{\delta \rho}{3(\rho + P)}$$
 conserved

(ii) energy conservation for non-interacting perfect fluids:

$$H^{-1}\dot{\rho}_i = -3(\rho_i + P_i)$$
 where $P_i = P_i(\rho_i) \Rightarrow \zeta_i = \frac{\delta\rho_i}{3(\rho_i + P_i)}$

(iii) conserved particle/quantum numbers (e.g., B, B-L,...)

$$H^{-1}\dot{n}_i = -3n_i \implies \zeta_i = \frac{\delta n_i}{3n_i}$$

microwave background signatures:

$$C_{I} = A^{2} \times$$
 $+ B^{2} \times$
 $+ 2 A B \cos \Delta \times$

adiabatic

CDM isocurvature

 $C_{I} = A^{2} \times$
 $+ 2 A B \cos \Delta \times$

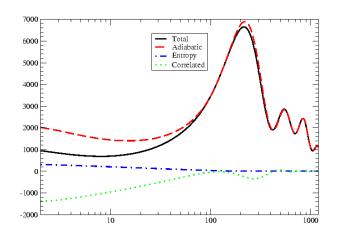
Bucher, Moodley & Turok '00 $n_S = 1$ Trotta, Riazuelo & Durrer '01

Amendola, Gordon, Wands & Sasaki '01

best-fit to Boomerang, Maxima & DASI

B/A = 0.3,
$$\cos \Delta = +1$$
, $n_S = 0.8$

$$\omega_{\rm b} = 0.02, \ \omega_{\rm cdm} = 0.1, \ \Omega_{\Lambda} = 0.7$$



Portsmouth

- o island city on the south coast of England
- historic home of the Royal Navy
- University of Portsmouth founded 1992
- Institute of Cosmology and Gravitation established in 2002
- 60+ researchers (academic staff, postdoctoral researchers, phd students and visiting researchers)





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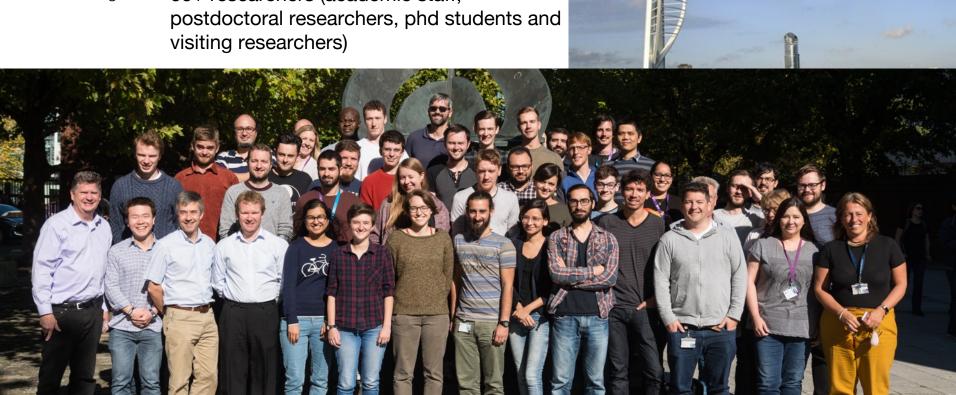
 60+ researchers (academic staff, postdoctoral researchers, phd students and visiting researchers)





Portsmouth

- island city on the south coast of England 0
- historic home of the Royal Navy 0
- University of Portsmouth founded 1992 \bigcirc
- Institute of Cosmology and Gravitation 0 established in 2002
- 60+ researchers (academic staff, 0 visiting researchers)



Einstein equations in an arbitrary gauge

Evolution equations

trace and trace-free spatial part of Einstein equations:

$$C'' + 2\mathcal{H}C' - \mathcal{H}A' - (2\mathcal{H}' + \mathcal{H}^2)A = -4\pi Ga^2 \left(\delta P + \frac{2}{3}\nabla^2\Pi\right),$$

$$\sigma' + 2\mathcal{H}\sigma - A - C = 8\pi Ga^2\Pi,$$

Energy+momentum constraints

time-time and time-space components:

$$3\mathcal{H}(C'-\mathcal{H}A) - \nabla^2(C-\mathcal{H}\sigma) = 4\pi G a^2 \delta \rho ,$$

 $C'-\mathcal{H}A = 4\pi G a^2 (\rho + P)(v+B) .$

Energy+momentum conservation

Fluid continuity and Euler equations:

$$\delta \rho' + 3\mathcal{H}(\delta \rho + \delta P) + 3(\rho + P)C' + (\rho + P)\nabla^{2}(v + E') = 0$$

$$(v + B)' + (1 - 3c_{s}^{2})\mathcal{H}(v + B) + A + \frac{1}{\rho + P}\left(\delta P + \frac{2}{3}\nabla^{2}\Pi\right) = 0$$

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Homogeneous cosmology

erturbation eory

calar perturbations

Vector perturba

Tensor pertur

Metric

erturbations

Geometrical interpretation

articular gauges

Conformal Newtonian/Longitud gauge

Iniform-density gauge quating gauge-invarian

Einstein equations

Einstein equations in an arbitrary gauge

Recovering Newton equations

Redshift-space disto

covering Newton d equations

Constraint equations in conformal Newtonian gauge

$$A=\Psi\,,\quad B=0\quad, C=\Phi\,,\quad E=0$$

Energy and momentum constraint equations

$$3\mathcal{H}(\Phi' - \mathcal{H}\Psi) - \nabla^2 \Phi = 4\pi G a^2 \delta \rho ,$$

$$\Phi' - \mathcal{H}\Psi = -4\pi G a^2 (\rho + P) V .$$

eliminate $\Phi' - \mathcal{H}\Psi$ gives *Poisson equation*:

$$\frac{\nabla^2}{a^2}\Phi = -4\pi G\delta\rho_c \; ,$$

where gauge-invariant comoving energy density (i.e., $\delta \rho$ in v+B=0 gauge)

$$\delta \rho_c \equiv \delta \rho + 3\mathcal{H}(\rho + P)V$$

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tomogeneous cosmology

'erturbation heory

Scalar perturbations

Power spectrum

Vector perturbation Tensor perturbation

Metric

perturbation

eometrical interpretation

articular gauges

Conformal Newtonian/Long gauge

Jniform-density gauge quating gauge-invarian

instein equations

Linstelli equations

Einstein equations in an arbitrary gauge

Recovering Newtonia equations

Redshift-space distor

covering Newtonian



Fluid equations in arbitrary gauge

Fluid continuity and Euler equations:

$$\delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) + 3(\rho + P)C' + (\rho + P)\nabla^2(v + E') = 0,$$

$$(v + B)' + (1 - 3c_s^2)\mathcal{H}(v + B) + A + \frac{1}{\rho + P}\left(\delta P + \frac{2}{3}\nabla^2\Pi\right) = 0.$$
Perturbation theory

For zero pressure perturbations (e.g., ACDM)

$$\delta \rho' + 3\mathcal{H}\delta \rho + 3\rho C' + \rho \nabla^2 (v + E') = 0,$$

$$(v + B)' + \mathcal{H}(v + B) + A = 0.$$

In comoving gauge (v + B = 0)

$$\delta \rho_c' + 3\mathcal{H}\delta \rho_c + 3\rho C_c' + \rho \nabla^2 V = 0,$$

$$A_c = 0.$$

and momentum constraint then reduces to $C_c'=0$. In conformal Newtonian gauge (E = B = 0) Euler equation becomes

$$V' + \mathcal{H}V + \Psi = 0.$$

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Recovering Newtonian fluid equations

Changing to density contrast

$$\delta \equiv \frac{\delta \rho}{\rho}$$

we then have for pressureless matter

▶ Poisson equation for conformal Newtonian potential:

$$\nabla^2 \Phi = -\frac{3}{2} \mathcal{H}^2 \delta_c$$

Continuity equation for comoving density:

$$\delta_c' + \nabla^2 V = 0$$

 \blacktriangleright Euler equation for conformal Newtonian velocity ($\vec{V}=\vec{\nabla}V$):

$$\vec{V}' + \mathcal{H}\vec{V} = -\vec{\nabla}\Psi$$

Coincide exactly to first-order Newtonian perturbation equations in an expanding cosmology, using $\Psi = -\Phi$ for zero anisotropic stress.

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Perturbation heory

Scalar perturb:

Fourier transform

ector perturbation

Metric .

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Geometrical interpreta

Gauge dependence

Conformal Newtonian/Longi

Newtonian/Long gauge

Uniform-density gauge quating gauge-invariant ariables

Einstein equations

arbitrary gauge
Recovering Newtonian fluid

equations

Recovering Newtonian

Redshift-space distortions

Continuity equation for comoving density contrast:

$$\delta_c' + \nabla^2 V = 0$$

divergence of peculiar velocities, $\theta \equiv \nabla^2 V$, seen in galaxy redshift surveys

$$\langle \theta^2 \rangle = f^2 \langle \delta_c^2 \rangle$$

Important probe of growth of structure given by $f\sigma_8$, where σ_8^2 is the variance of the matter power spectrum on 8 Mpc scales (\approx 1).

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erturbation

theory

Fourier transforms Power spectrum

Vector perturbation

Metric

erturbations

eometrical interpretation

Gauge dependence

Conformal Newtonian/Long

Newtonian/Long gauge

Uniform-density ga quating gauge-inva

notoin oquations

Einstein equations

Einstein equations in an arbitrary gauge Recovering Newtonian fluid

Redshift-space distortions

Recovering Newton fluid equations



Second-order equation for δ

Continuity equation for comoving density contrast:

$$\delta_c' + \nabla^2 V = 0$$

Taking time derivative

$$\delta_c'' + \nabla^2 V' = 0$$

plus Euler equation

$$V' + \mathcal{H}V = -\Psi$$

eliminating V' and V gives

$$\delta_c'' + \mathcal{H}\delta_c' - \nabla^2 \Psi = 0$$

Using $\Psi = -\Phi$ and Poisson equation

$$\nabla^2 \Phi = -\frac{3}{2} \mathcal{H}^2 \delta_c$$

gives linear second-order differential equation

$$\delta_c'' + \mathcal{H}\delta_c' - \frac{3}{2}\mathcal{H}^2\delta_c = 0$$

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Scalar perturbati

Fourier transforms

Vector perturbation

Motric

perturba:

eometrical interpretat

uge dependence irticular gauges

Conformal

Newtonian/Lon gauge

Uniform-density g

Einstein equations

arbitrary gauge
Recovering Newtonian fluid

Redshift-space distortions

Recovering Newtor fluid equations

Growth of structure

In absence of any pressure perturbations, e.g., ACDM

$$\delta_c'' + \mathcal{H}\delta_c' - \frac{3}{2}\mathcal{H}^2\delta_c = 0$$

Growth of structure is independent of scale in Λ CDM cosmology.

$$\delta_k(\tau) = D_+(\tau)\delta_{k,0} + D_-(\tau)\tilde{\delta}_{k,0}$$

- Growing mode $D_+(au)$ normalised so that $D_+(au_0)=1$ today
- ▶ Decaying mode $D_{-}(\tau)$ assumed negligible at late times
- lacksquare In matter-dominated cosmology $(\Omega_m=1)$ then $D_+\propto a$

$$f \equiv \frac{d \ln D_+}{d \ln a} = 1$$

In ΛCDM

$$f\simeq\Omega_m^{6/11}$$

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Homogeneous cosmology

Perturbat heory

Scalar perturbation

Power spectrum Vector perturbations

Metric

perturbations

Geometrical interpretation

Particular gauges
Conformal

Newtonian/Longitudin gauge Uniform-density gauge

quating gauge-invaria ariables

Einstein equations

Einstein equations in an arbitrary gauge Recovering Newtonian fluid equations

. Redshift-space dist

Recovering Newtonian fluid equations