5th Tah Poe School on Cosmology July 2019

Theoretical cosmology

David Wands
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Theoretical cosmology

Introduction to Inflation - part IV
Models of single-field slow-roll inflation

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Models of inflation
Large-field inflation
Small-field inflation
Natural inflation
Higgs inflation
Starobinsky inflation
Inflationary phenomenolgy
Multi-field models and non-Gaussianity
Stochastic inflation

previously...

- Primordial perturbations from inflation:
  - initial conditions from sub-Hubble scale vacuum fluctuations
  - gauge-invariant field+metric fluctuations from inflaton field fluctuations
  - slow-roll induces weak scale-dependence
    - *confirmed by observations*
  - gravitational waves (tensors) also predicted
    - *but not yet seen*
- Observations can
  - constrain model parameters
  - discriminate between different kinds of inflation models

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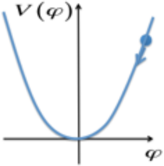
Models of inflation: large-field inflation

- also known as chaotic or monomial inflation

$$V(\varphi) = \frac{\lambda_p}{p!} \frac{\varphi^p}{M_P^{p-4}}$$



- slow-roll parameters:

$$\epsilon_V = \frac{p^2}{16\pi}\left(\frac{M_P}{\varphi}\right)^2 \quad , \quad \eta_V = \frac{p(p-1)}{8\pi}\left(\frac{M_P}{\varphi}\right)^2$$

- slow-roll, $\epsilon, |\eta| \ll 1$, requires $\varphi \gg pM_P$

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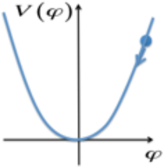

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- slow-roll, $\epsilon, |\eta| \ll 1$, requires $\varphi \gg pM_P$
- e-folds:

$$N(\varphi, \varphi_{\rm end}) \simeq \int_{\varphi_{\rm end}}^{\varphi} \sqrt{\frac{4\pi}{\epsilon_V}} \frac{d\varphi}{M_P} \simeq \int_{\varphi_{\rm end}}^{\varphi} \frac{8\pi}{p} \frac{\varphi}{M_P} \frac{d\varphi}{M_P}$$

$$\Rightarrow \quad \varphi(N) \simeq \sqrt{\frac{pN}{4\pi}}\, M_P$$

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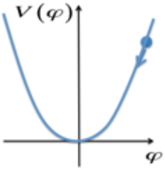

- slow-roll solution $\varphi(N)$:

$$\epsilon \simeq \frac{p}{4N} \quad , \quad \eta \simeq \frac{p-1}{2N}$$

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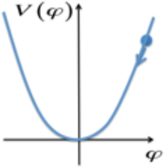

- slow-roll solution $\varphi(N)$:

$$\epsilon \simeq \frac{p}{4N} \quad , \quad \eta \simeq \frac{p-1}{2N}$$

- e.g., $p=2$, quadratic inflaton, $V = m^2\varphi^2/2$

$$\epsilon \simeq \eta \simeq \frac{1}{2N} \simeq 0.01 \quad \text{for } N = 50$$

$$n_R - 1 \simeq -0.04 \quad , \quad r \simeq 0.16$$

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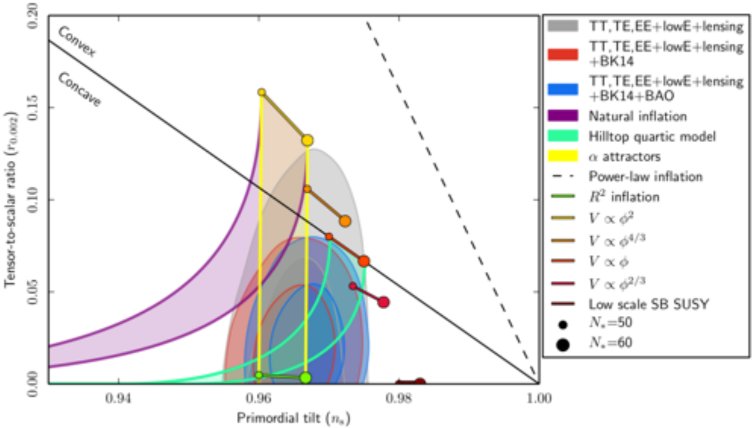
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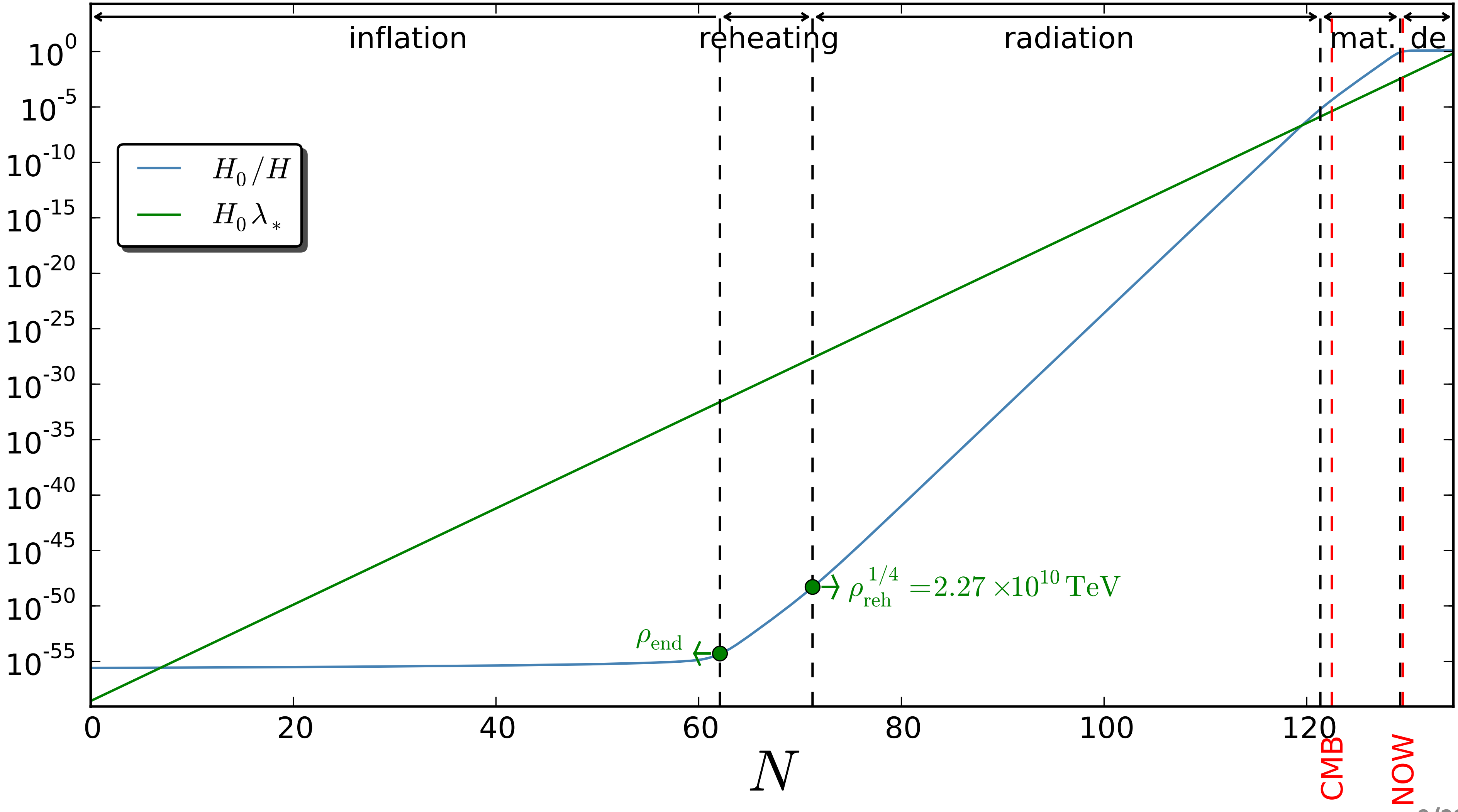
N_*=50
N_*=60

Role of Reheating



Reheating: simplest case

- Coherent inflaton field oscillations are non-relativistic inflaton particles

$$\langle \rho_\varphi \rangle_t = m n_\varphi \propto a^{-3}$$

- Perturbative decay ($\Gamma < m$) to light (relativistic) particles

$$\mathcal{L}_{\rm int} = -\lambda_i \sigma \varphi \chi_i^2 - \lambda_j \varphi \bar{\psi}_j \psi_j$$

$$\Rightarrow \quad \Gamma = \frac{\lambda_i^2 \sigma^2}{8\pi m} + \frac{\lambda_j^2 m}{8\pi}$$

$$\dot{\rho}_\varphi + 3H(\rho_\varphi + P_\varphi) = -\Gamma \varphi^2$$

(non-perturbative decay (preheating) can also be important)

- energy transfered from inflaton to radiation when $H < \Gamma$

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Reheating: temperature and e-folds

- decay followed by thermalisation to reheat temperature

$$\rho_{\rm rh} \leq \frac{3H^2_{\rm decay}}{8\pi G} = \frac{3\Gamma^2}{8\pi G}$$

$$\Rightarrow \quad T_{\rm rh} \leq 0.2 \left(\frac{100}{g_{\rm rh}}\right)^{1/4} (\Gamma M_P)^{1/2}$$

- duration of reheating (and effective equation of state) affects expansion history after inflation and hence the number of e-folds when present horizon scale exits Hubble scale during inflation

$$N_* = 67 - \ln\left(\frac{k}{H_0}\right) + \frac{1}{4}\ln\left(\frac{V_*^2}{M_P^4 \rho_{\rm end}}\right) + \frac{1}{12}\ln\left(\frac{\rho_{\rm rh}}{\rho_{\rm end}}\right) - \frac{1}{12}\ln(g_{\rm rh})$$

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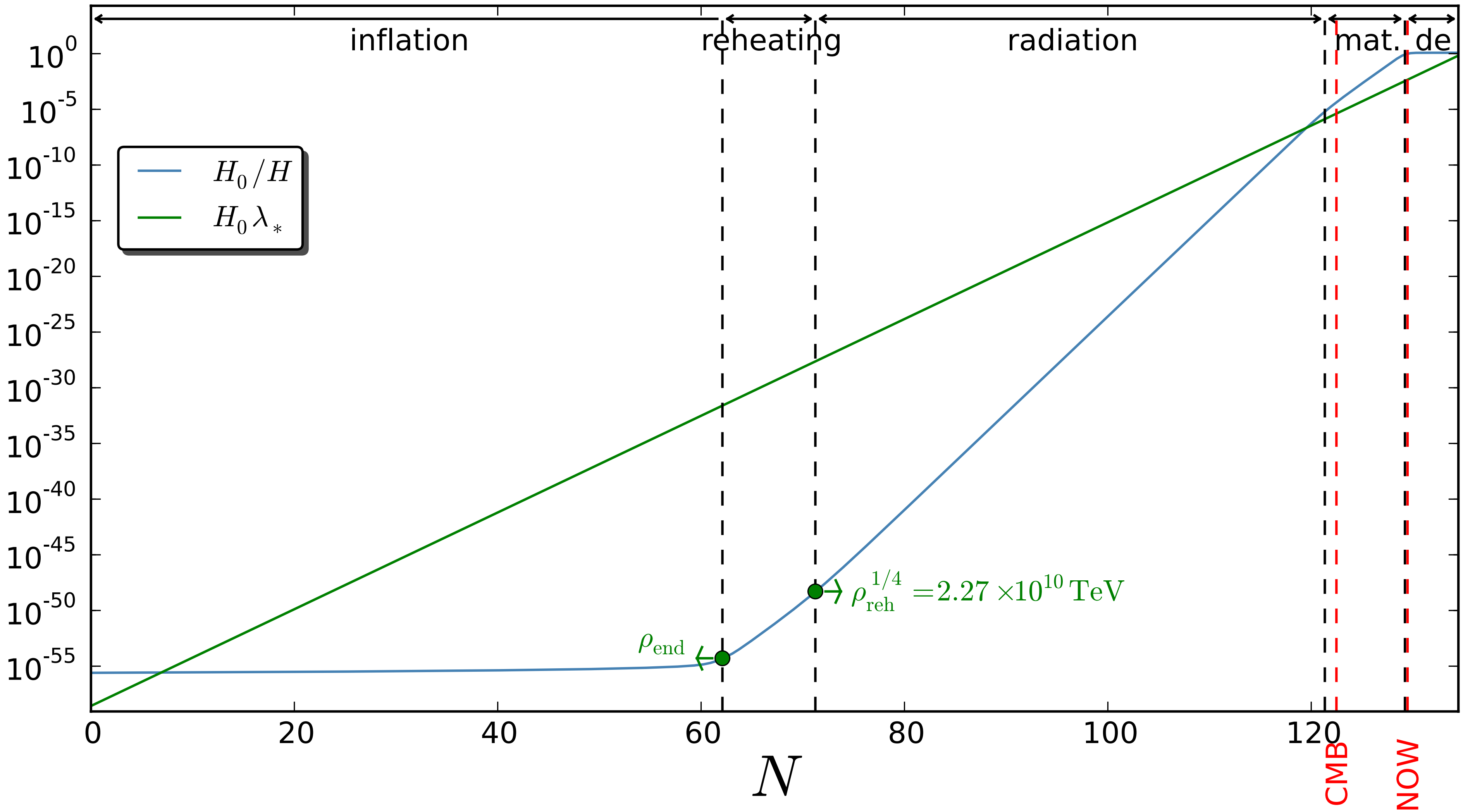

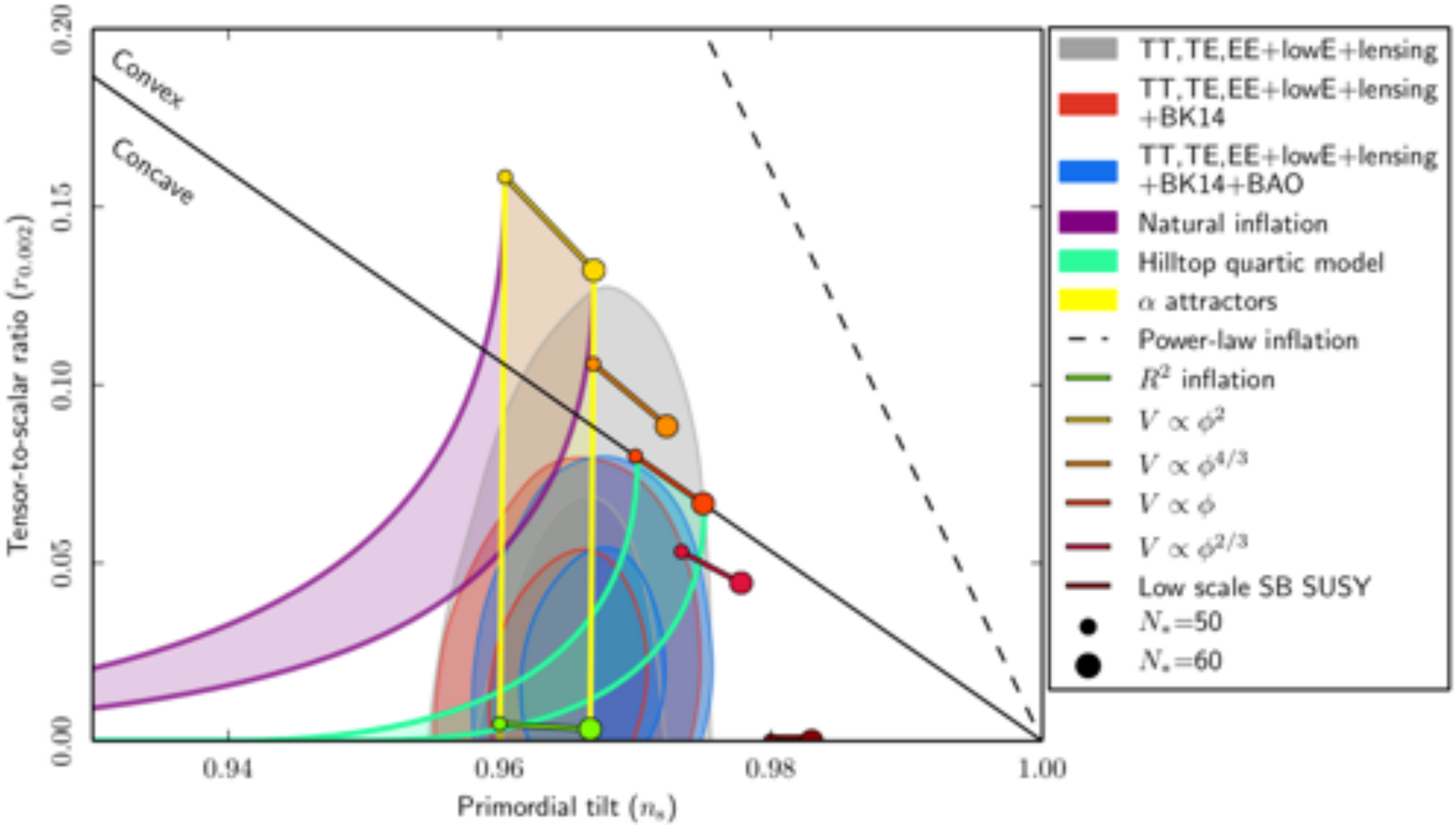
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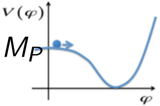
Models of inflation: small-field inflation



- also known as hill-top inflation: $\varphi < \mu < M_P$

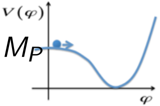
$$V(\varphi) = V_0 \left(1 - \left(\frac{\varphi}{\mu} \right)^p + \ldots \right) \quad \text{for } p \geq 2$$

- slow-roll parameters:

$$\epsilon \propto \left(\frac{M_P}{\mu} \right)^2 \left(\frac{\varphi}{\mu} \right)^{2(p-1)} \quad , \quad \eta \propto \left(\frac{M_P}{\mu} \right)^2 \left(\frac{\varphi}{\mu} \right)^{p-2}$$

$$\Rightarrow \quad \frac{\epsilon}{|\eta|} = \frac{p}{2(p-1)} \left(\frac{\varphi}{\mu} \right)^p \ll 1 \quad \text{for } \varphi \ll \mu$$

Models of inflation: small-field inflation



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- e-folds:

$$N(\varphi, \varphi_{\rm end}) \simeq \frac{8\pi}{p(p-2)} \left(\frac{\mu}{M_P}\right)^2 \left(\frac{\varphi}{\mu}\right)^{-(p-2)} \quad \text{for } p > 2$$

$$\Rightarrow \quad \eta(N) \simeq -\left(\frac{p-1}{p-2}\right)\frac{1}{N} \quad \text{for } N \gg 1$$

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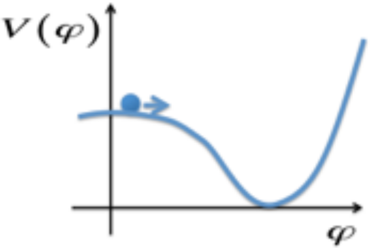

- initial conditions problem for small field inflation?
  - require $\varphi_{\rm ini} \ll \mu$ for $N \gg 1$
  - *how likely is this?*
  - need a theory of initial conditions for inflation
    - *stochastic inflation*

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Models of inflation: Natural inflation

- pseudo-Nambu Goldstone boson, weakly-broken $U(1)$ symmetry

$$V = \frac{V_0}{2}\left[1+\cos\left(\frac{\sqrt{2}\varphi}{\mu}\right)\right]$$

- corresponds to hill-top inflation with $n=2$:

$$\epsilon = \frac{1}{4\pi}\left(\frac{M_P}{\mu}\right)^2\left(\frac{\varphi}{\mu}\right)^2 \quad , \quad \eta \simeq -\frac{1}{4\pi}\left(\frac{M_P}{\mu}\right)^2$$

slow-roll inflation, $|\eta| \ll 1$, requires $\mu \gg M_P$.

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Models of inflation: Starobinsky's R^2 inflation

- $f(R)$ gravity:

$$f(R) = R + \frac{1}{6M^2} R^2$$

- conformal transformation to Einstein gravity

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = f'(R) g_{\mu\nu}$$

- potential for minimally-coupled field
$\chi \propto M_P \ln f'(R)$

$$V(\chi) = M^2 M_P^2 \left(1 - e^{-\sqrt{3/2}\chi/M_P}\right)^2$$

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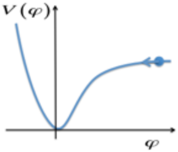


- plateau inflation:

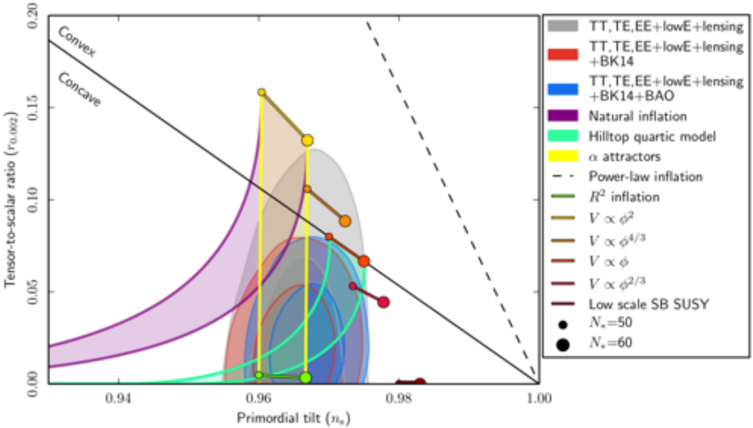
$$V(\chi) = M^2 M_P^2 \left(1 - e^{-\sqrt{3/2}\chi/M_P}\right)^2$$

- corresponds to "small-field" inflation in limit $p \to \infty$:

$$\eta \simeq -\frac{1}{N} \quad , \quad \epsilon \simeq \frac{3}{4N^2} \ll |\eta|$$

- observational parameters:

$$n_R - 1 \simeq -\frac{2}{N} \approx -0.04 \quad , \quad r \simeq \frac{12}{N^2} \approx 0.004$$



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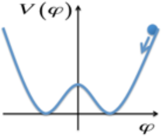
Models of inflation: Higgs inflation

- symmetry-breaking potential:

$$V(\varphi) = \frac{\lambda}{4}\left(\varphi^2 - \varphi_{vev}^2\right)^2$$



- requires large non-minimal coupling $\xi \gg 1$ to curvature:

$$\mathcal{L}_{\rm nmc} = \xi \varphi^2 R$$

- conformal transformation to minimally-coupled field $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = (\xi\varphi^2/M_P^2) g_{\mu\nu}$

$$V \to \tilde{V}(\psi) = \frac{\lambda}{4\xi^2} M_P^4 \left(1 - e^{-\psi/M_P} + \ldots\right)$$

- corresponds to small field inflation in limit $p \to \infty$:

$$\eta \simeq -\frac{1}{N} \quad , \quad \epsilon \sim \eta^2$$

- predictions very similar to Starobinsky's R^2-inflation

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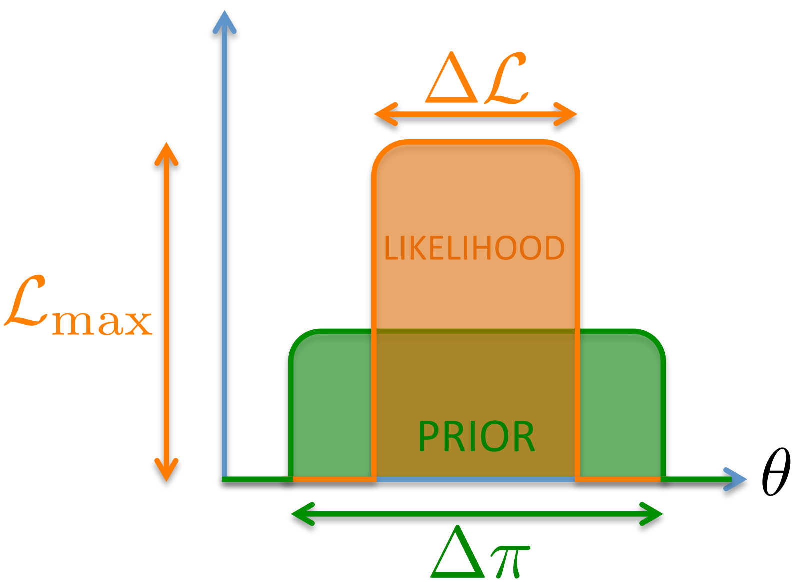
Bayesian Approach

to model comparison

Bayesian evidence: Integral of the likelihood over parameter prior



$$\mathcal{E}(\mathcal{M}) = \mathcal{L}_{\max} \frac{\Delta\mathcal{L}}{\Delta\pi}$$

Compromise between quality of fit and simplicity

Bayes factor = ratio of evidence

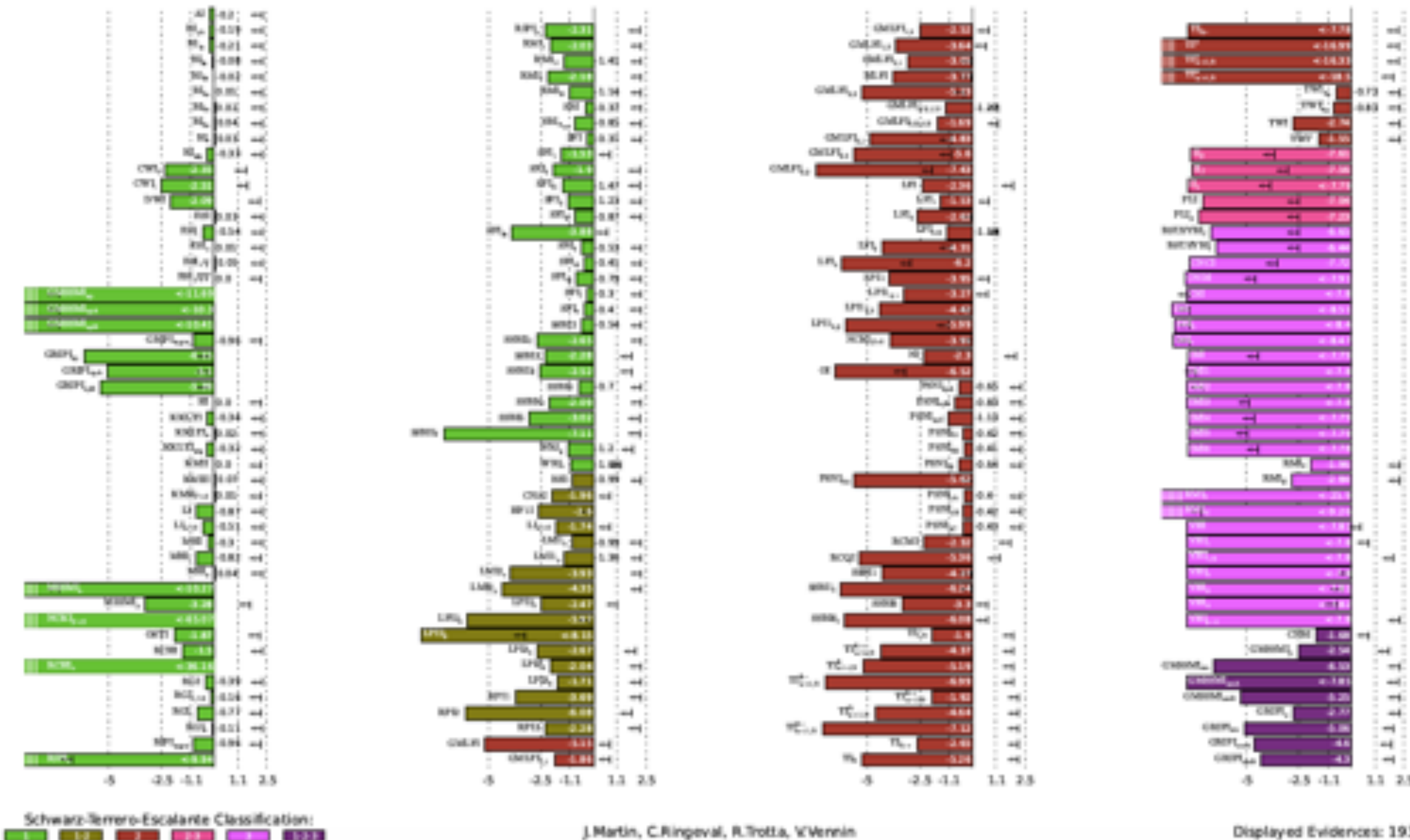
$$B_{ij} = E(M_i) / E(M_j)$$

Jeffreys scale

• Strong evidence	$\ln(B_{ij}) > 5$
• Moderate evidence	$\ln(B_{ij}) > 2.5$
• Weak evidence	$\ln(B_{ij}) > 1$
• Inconclusive	$\ln(B_{ij}) < 1$

Bayesian evidences computed with Planck

Martin, Ringeval, Trotta & Vennin (2014): 193 inflaton models

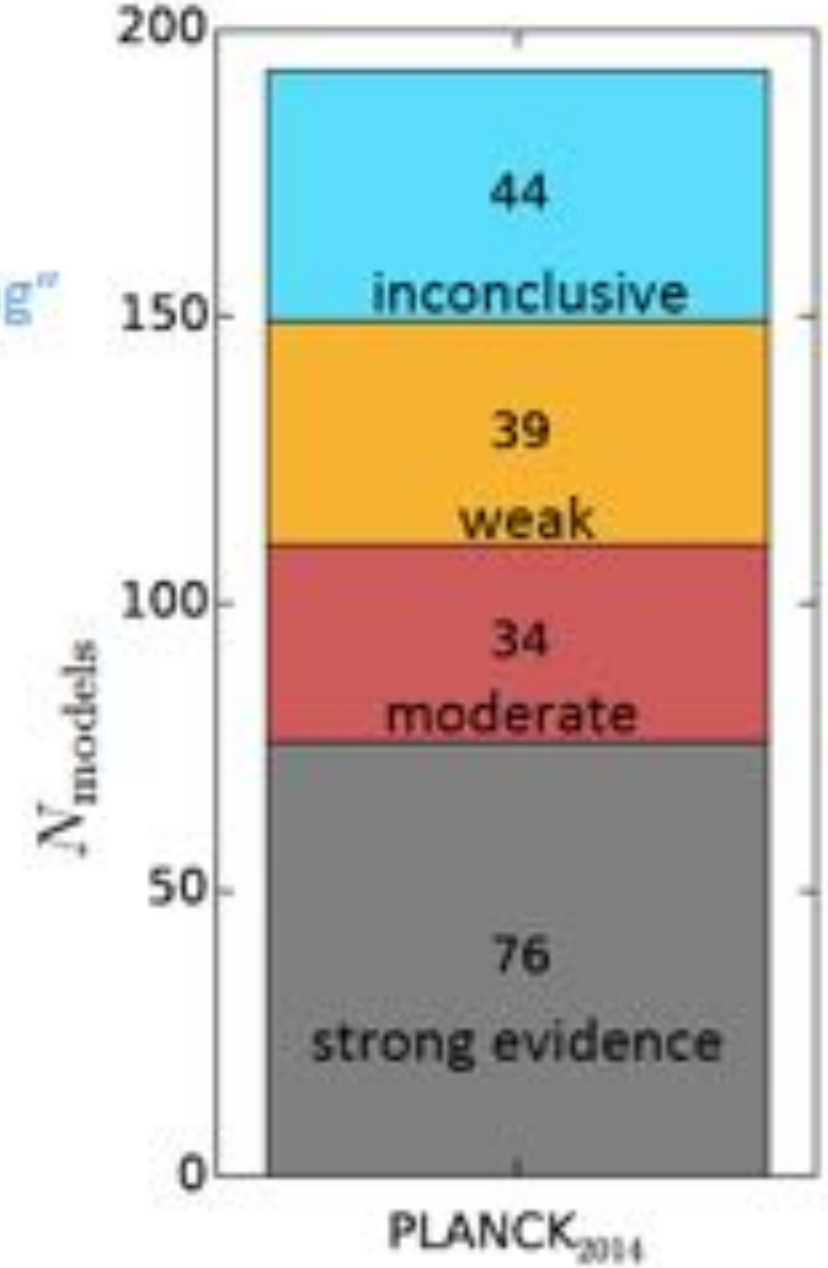


Bayesian evidence computed with Planck data

Summary of the results

- One third of the models are "ruled out"
  - $\ln (E / E_{HI}) < -5$
- Some of these models are killed by "fine-tuning"
- Planck favors "Plateau Inflation", e.g., Higgs or Starobinsky inflation

Model	$\Delta\chi^2$	$\ln B_{A,\mathrm{ref}}$
R^2 inflation	+0.8	0
Power-law potential $\phi^{2/3}$	+6.5	−2.4
Power-law potential ϕ^2	+8.6	−4.7
Power-law potential ϕ^4	+43.3	−23.3
Natural inflation	+7.2	−2.4
SUSY α-attractor	+0.7	−1.8



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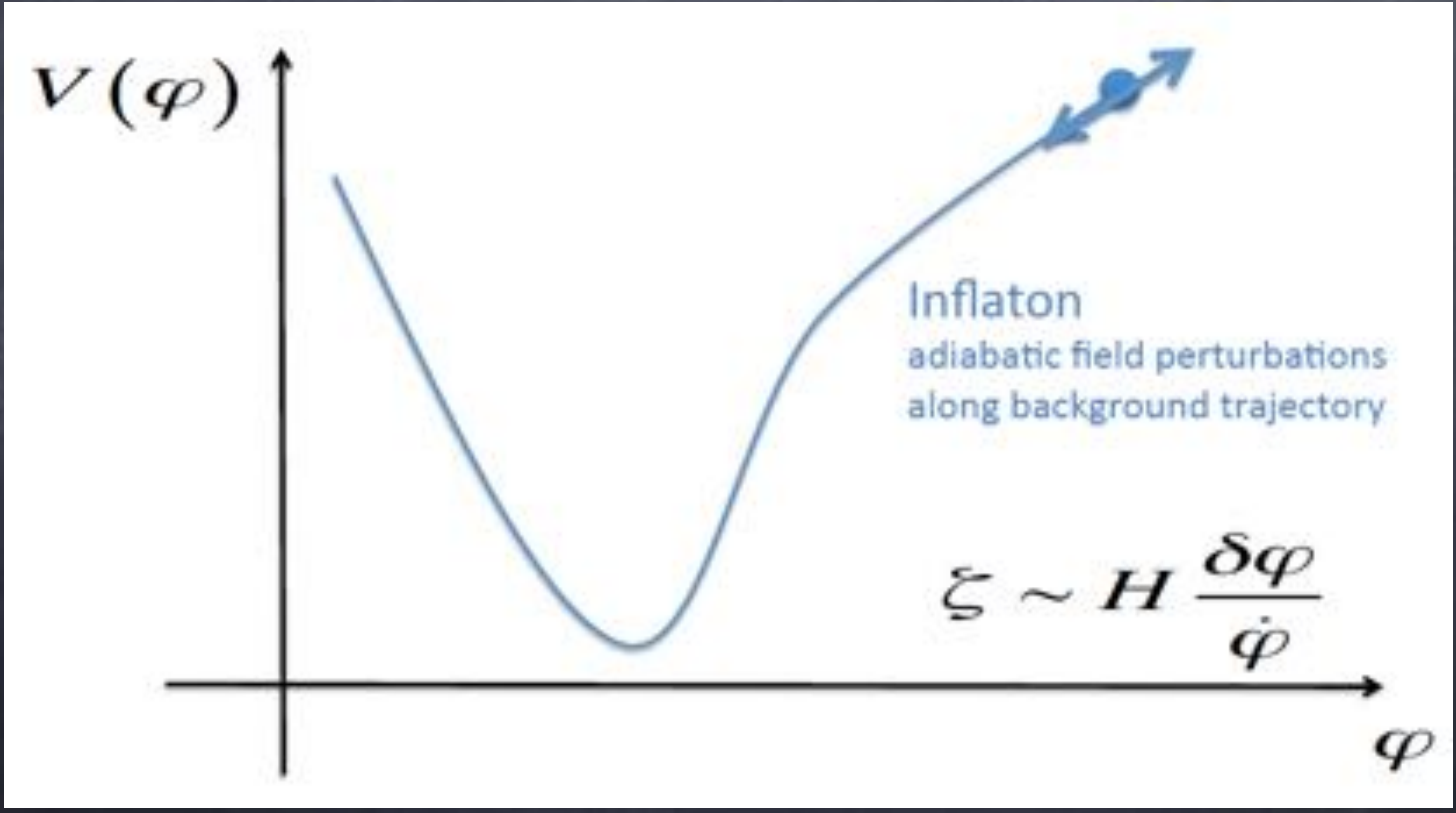
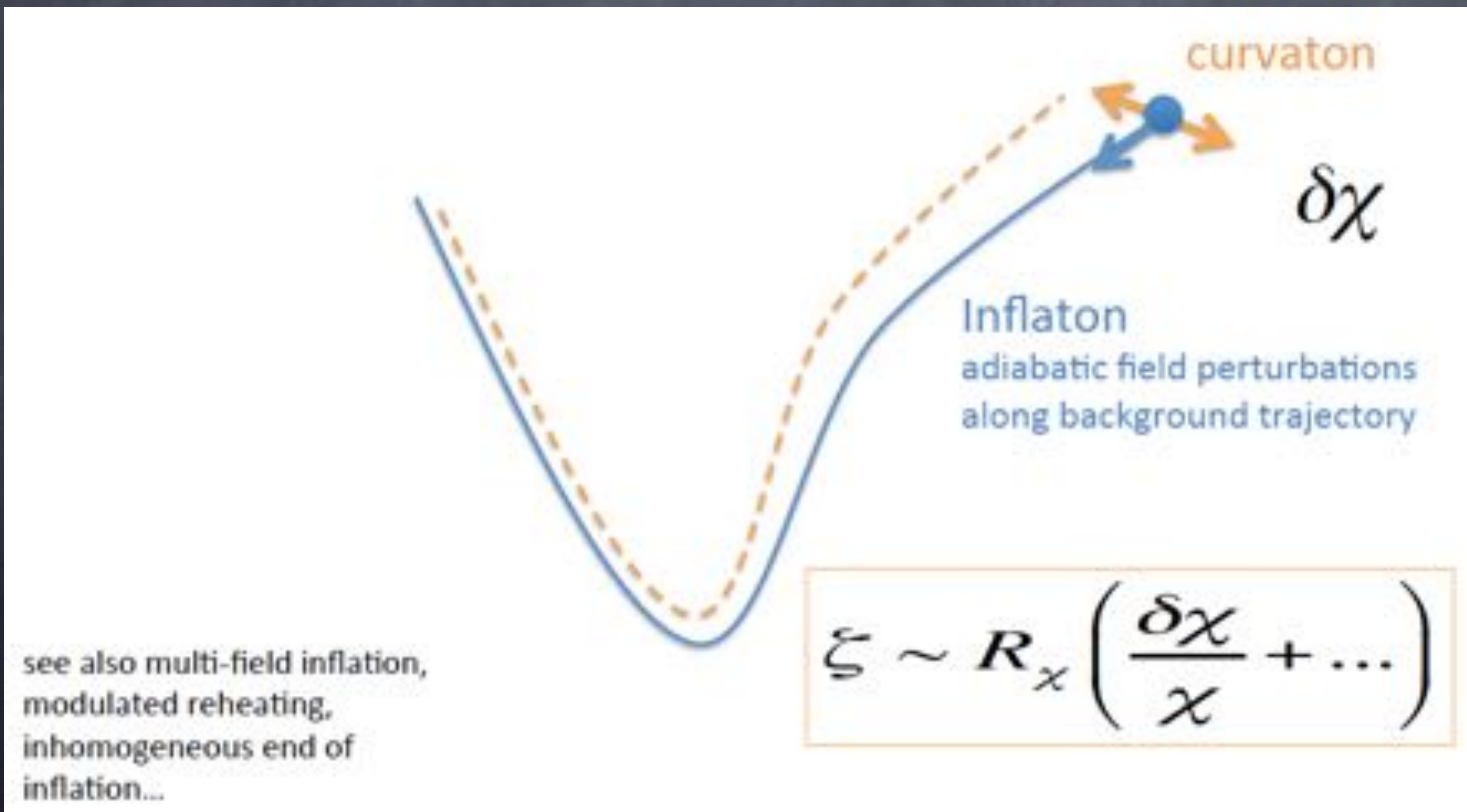
inflationary phenomenology

- Could inflation be very different from a simple single scalar field?
- and, if it was, how would we know?

single-field phenomenology



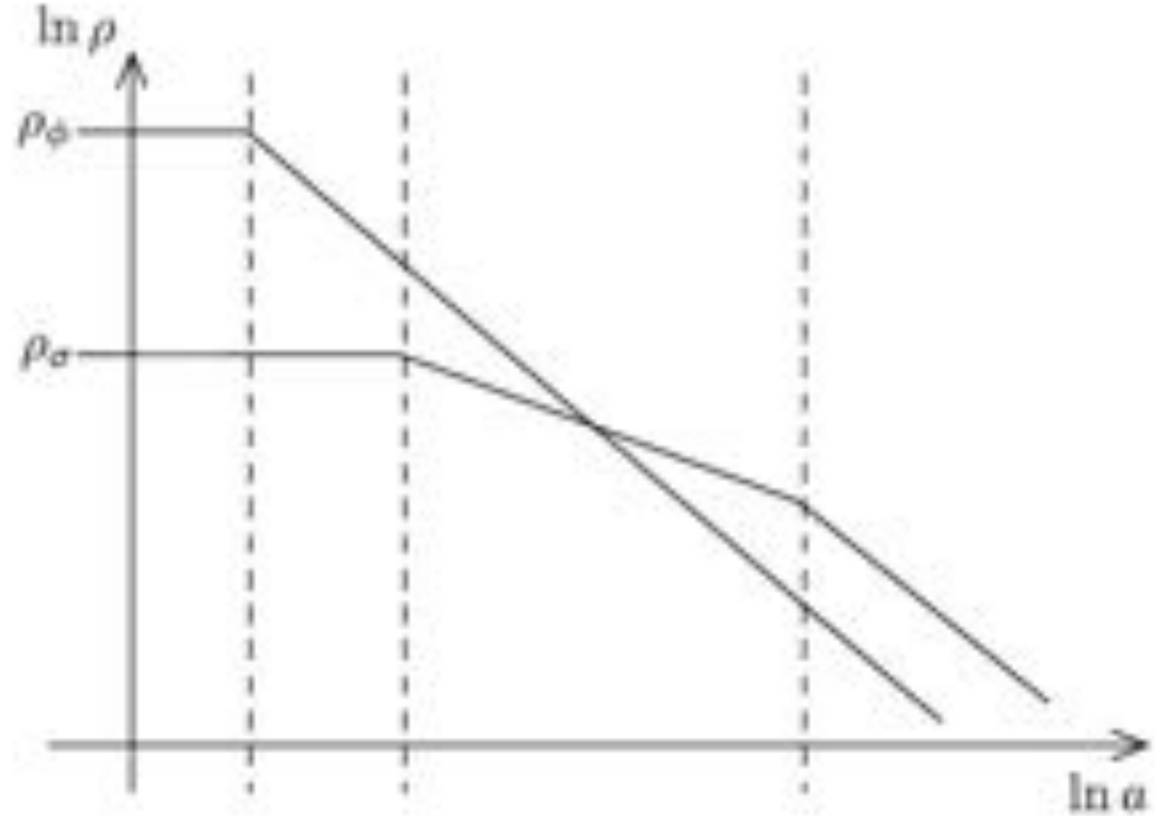
multi-field phenomenology



Inflaton models + curvaton field, χ

Curvaton scenarios with quadratic potential $V(\phi,\chi) = U(\phi) + m_\chi^2 \chi^2 / 2$

Linde and Mukhanov, 1997
Enqvist and Sloth, 2001
Lyth and Wands, 2001
Moroi and Takahashi, 2001
Langlois and Vernizzi, 2004

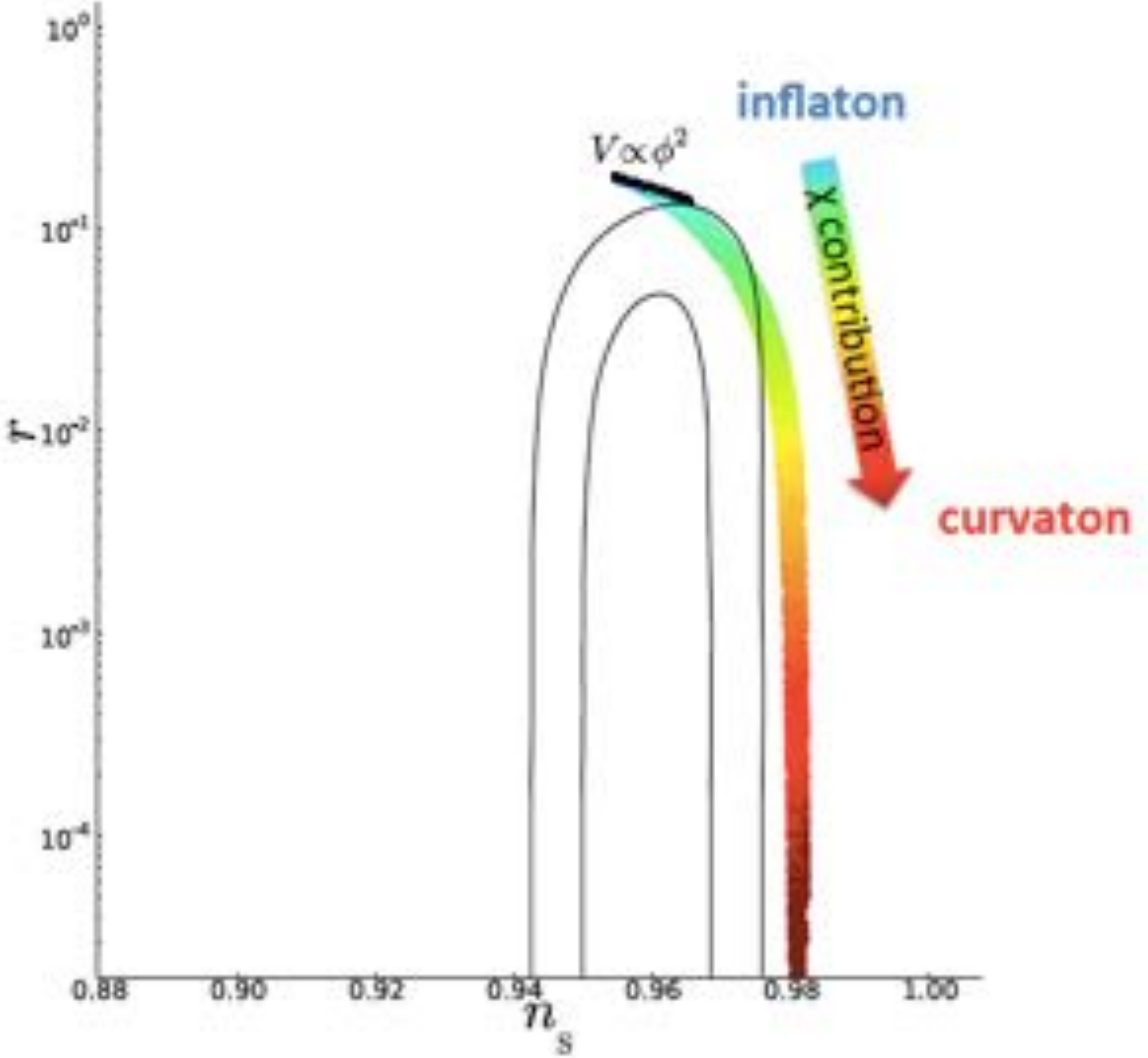


more reheating parameters: $\Gamma_\phi \rightarrow \Gamma_\phi\,,\ \Gamma_\chi\,,\ m_\chi\,,\ \chi_{\rm end}$

primordial perturbations directly dependent on reheating
(not just through the expansion N.)

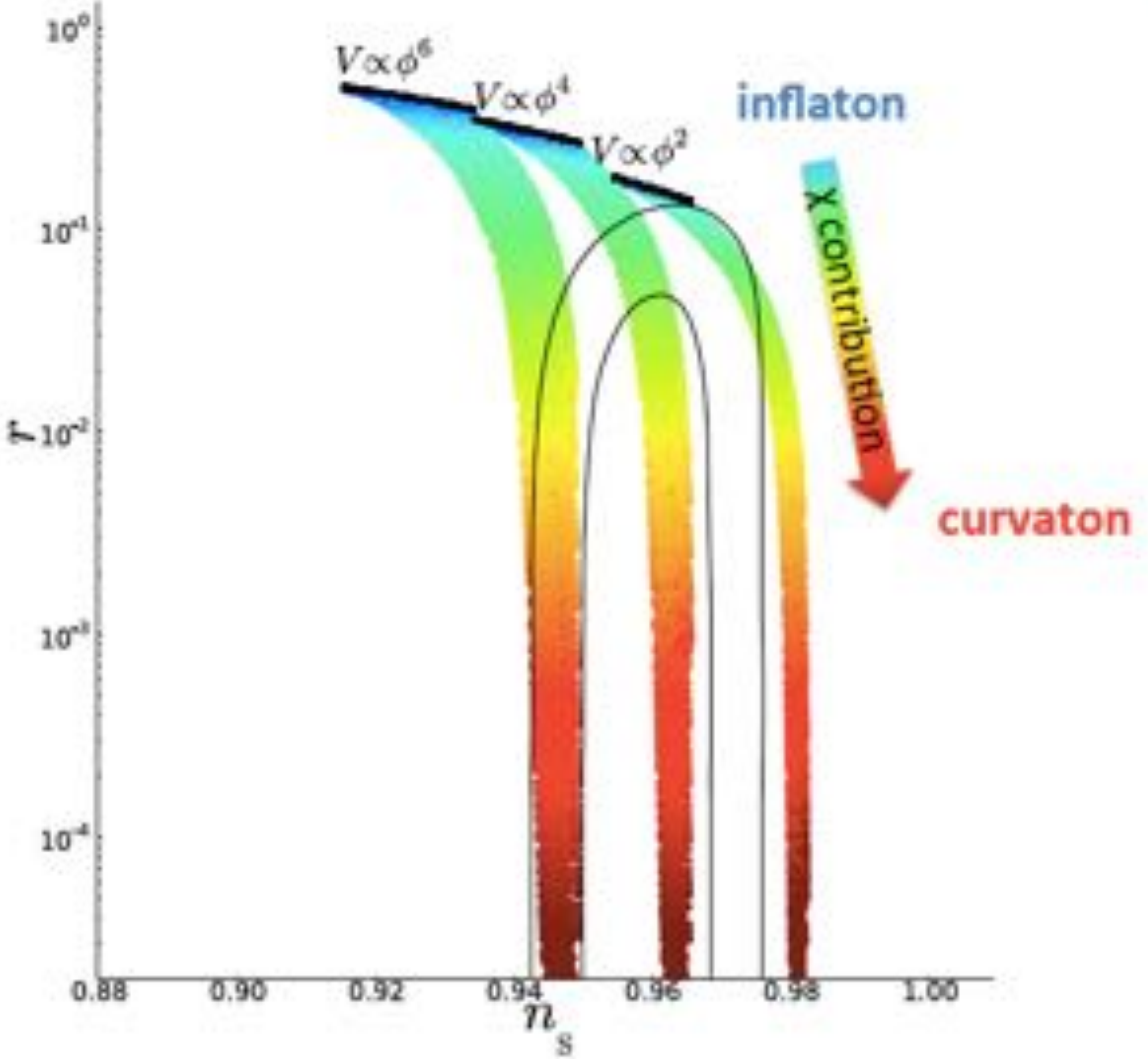
quadratic large-field (chaotic) inflaton plus quadratic curvaton, χ

Bartolo & Liddle (2002)
Ellis, Fairbairn & Sueiro (2014)
Byrnes, Cortes & Liddle (2014)
Smith & Grin (2015)
Vennin, Koyama & Wands (2015)



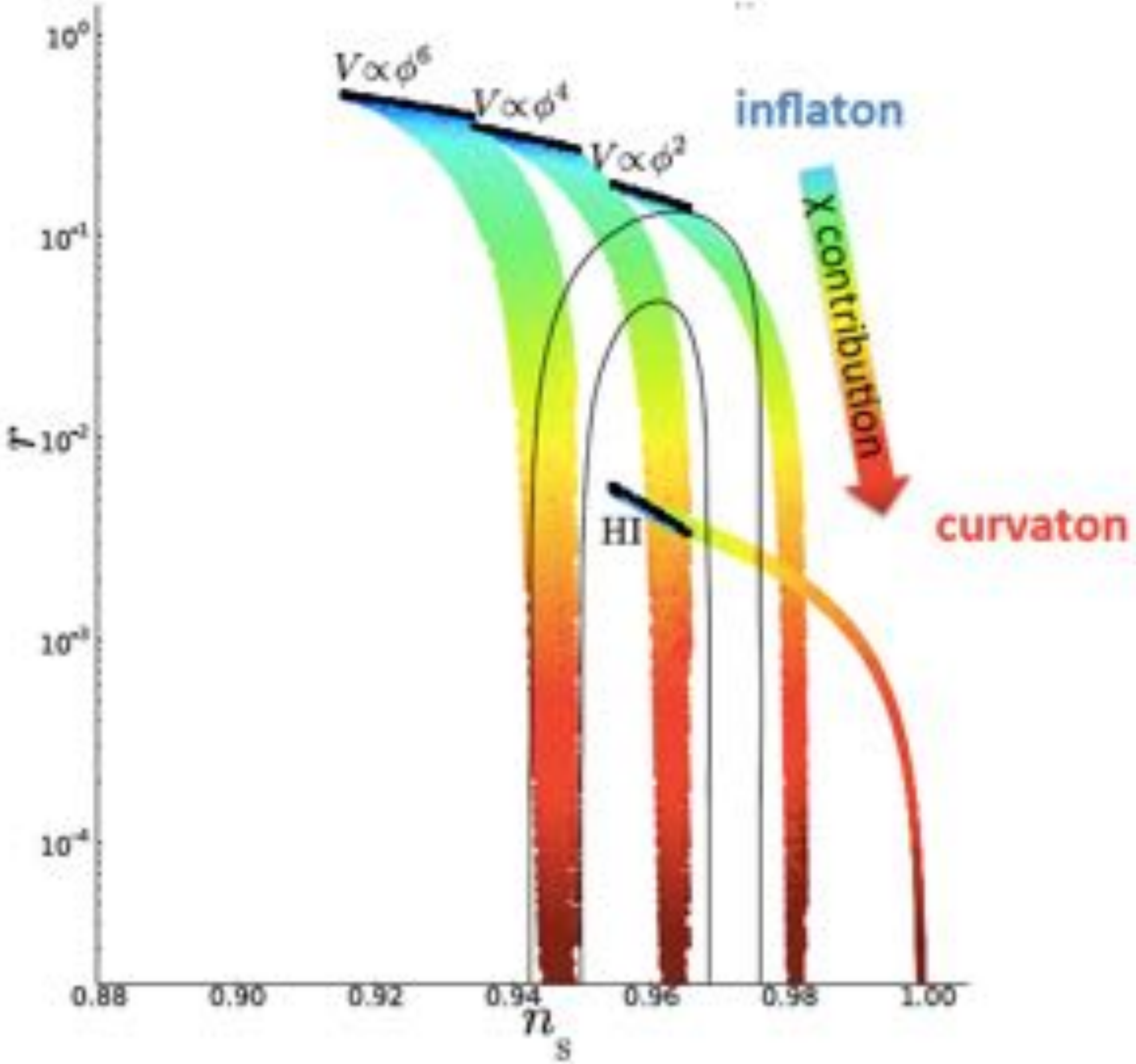
large-field inflaton (LFI) plus quadratic curvaton, χ

Vennin, Koyama and Wands (2015)



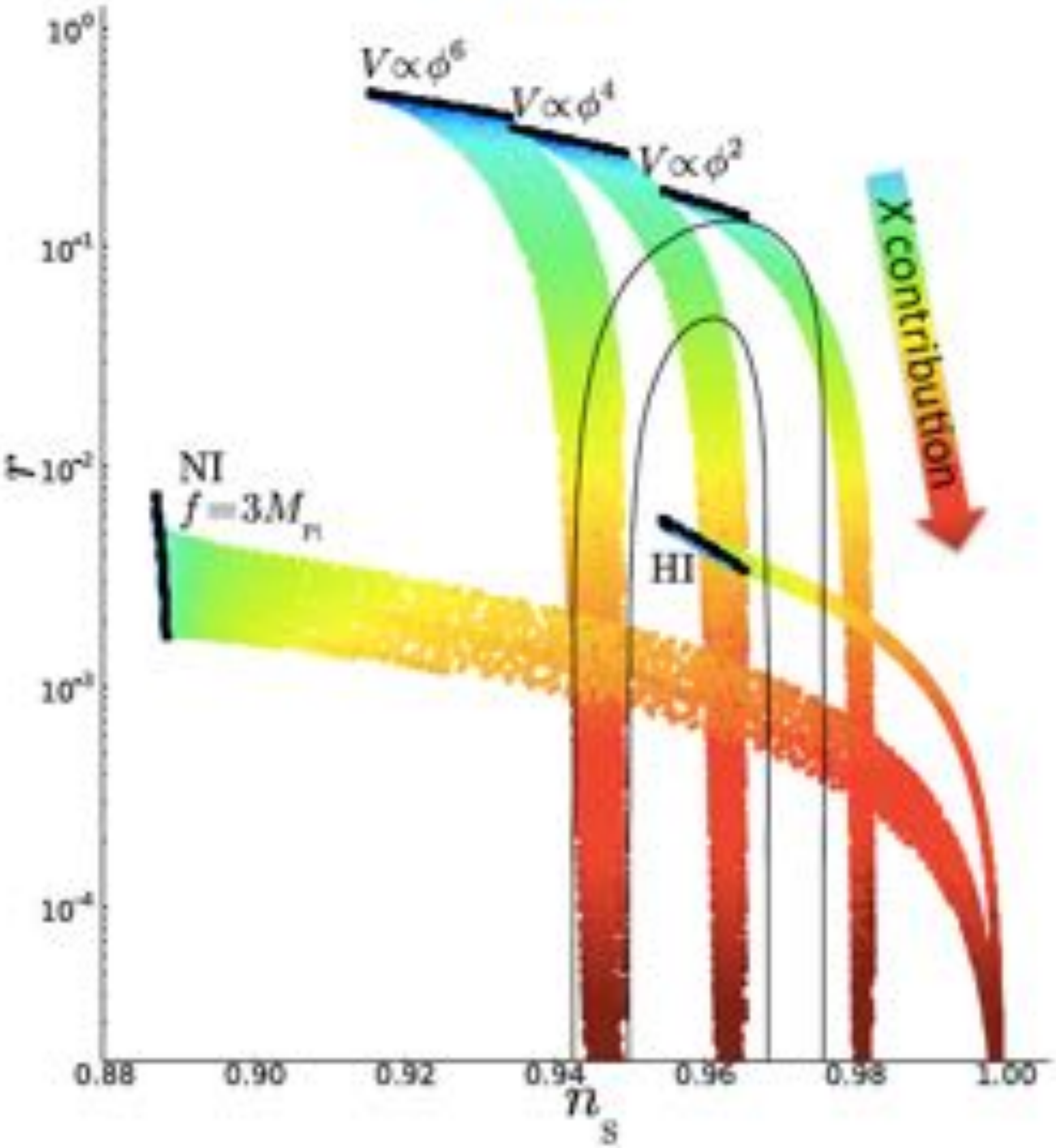
Higgs/Starobinsky inflation (HI) & LFI plus quadratic curvaton, χ

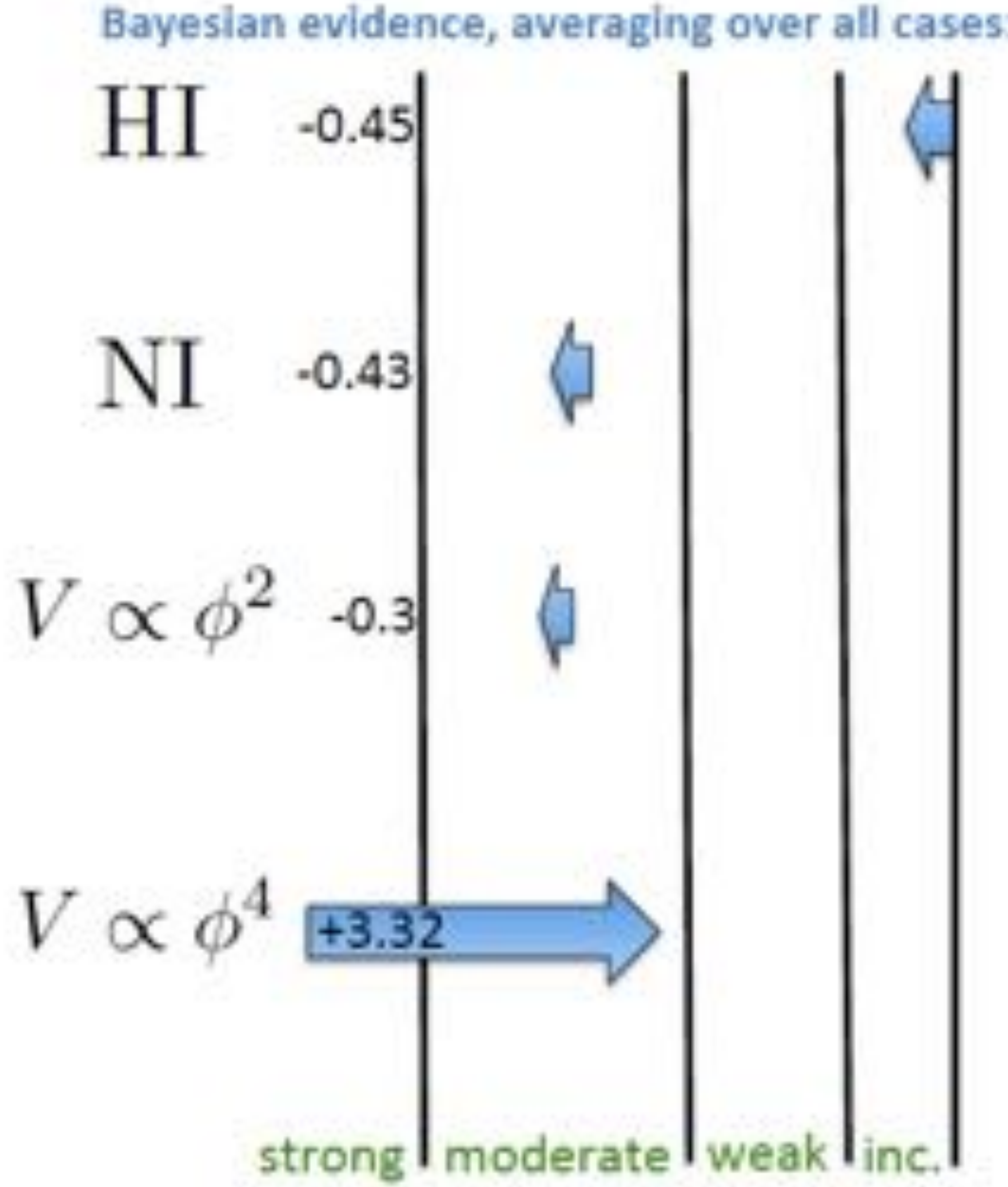
Vennin, Koyama and Wands (2015)



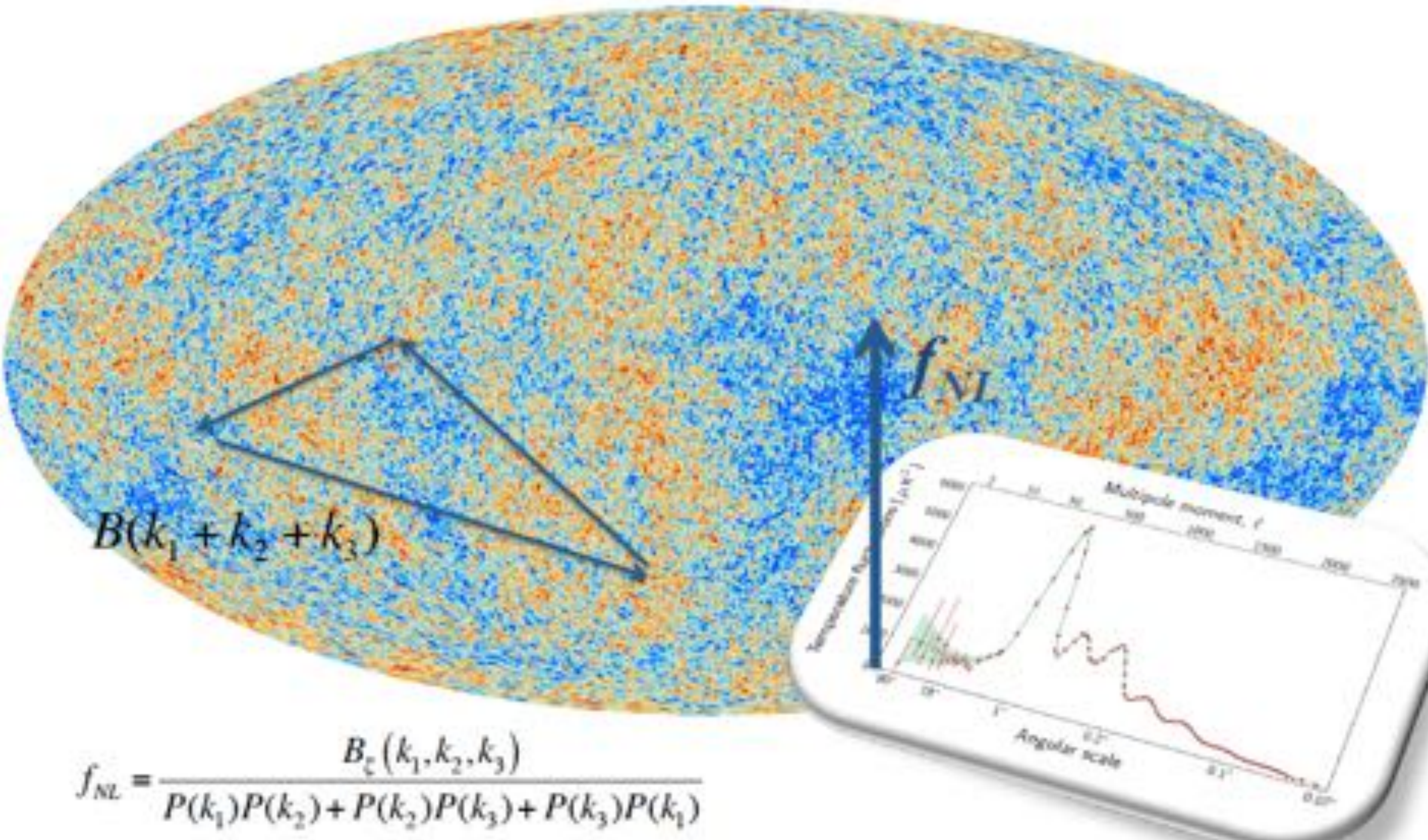
Add a weakly coupled scalar field, χ

Vennin, Koyama and Wands (2015)





more information in higher-order correlators...



$$f_{NL} = \frac{B_\zeta(k_1, k_2, k_3)}{P(k_1)P(k_2) + P(k_2)P(k_3) + P(k_3)P(k_1)}$$

curvaton scenario:

Linde & Mukhanov 1997; Enqvist & Sloth, Lyth & Wands, Moroi & Takahashi 2001

curvaton χ = weakly-coupled, late-decaying scalar field

- light field (m<H) during inflation acquires an almost scale-invariant, ***Gaussian distribution of field fluctuations*** on large scales
- **quadratic energy density** for free field, $\rho_\chi = m^2\chi^2/2$
- spectrum of initially isocurvature density perturbations

$$\zeta_\chi \approx \frac{1}{3}\frac{\delta\rho_\chi}{\rho_\chi} \approx \frac{1}{3}\left(\frac{2\chi\delta\chi + \delta\chi^2}{\chi^2}\right)$$

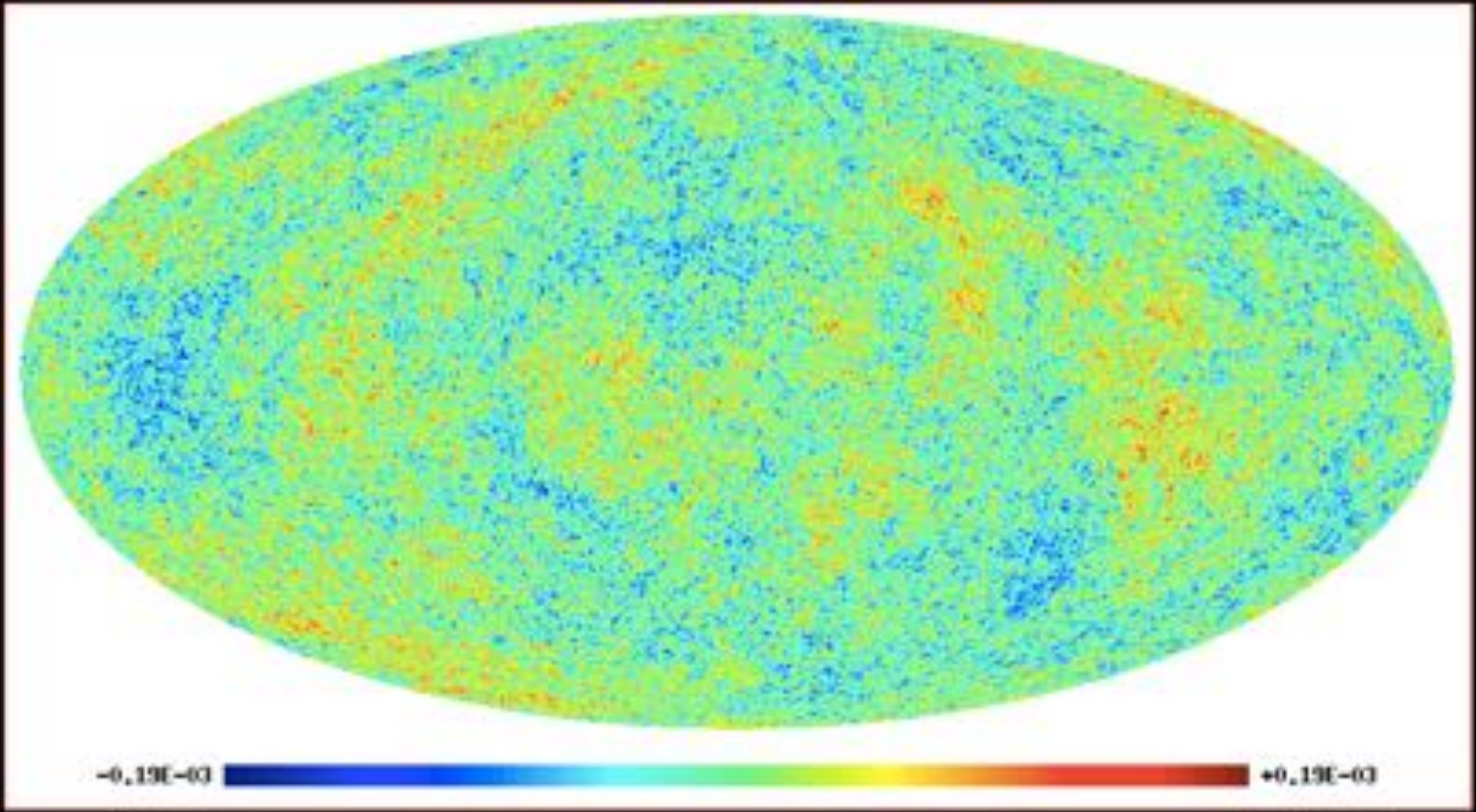
- **transferred to radiation when curvaton decays** after inflation with some **efficiency,** $0 < R_\chi < 1$, where $R_\chi \approx \Omega_{\chi,decay}$

$$\zeta = R_\chi \zeta_\chi \approx \frac{R_\chi}{3}\left(2\frac{\delta\chi}{\chi} + \frac{\delta\chi^2}{\chi^2}\right)$$

$$= \zeta_G + \frac{3}{4R_\chi}\zeta_G^2 \quad \Rightarrow \quad f_{NL} = 5/4R_\chi$$

Newtonian metric potential a ***Gaussian random field***

$\Phi(x) = \phi_G(x)$

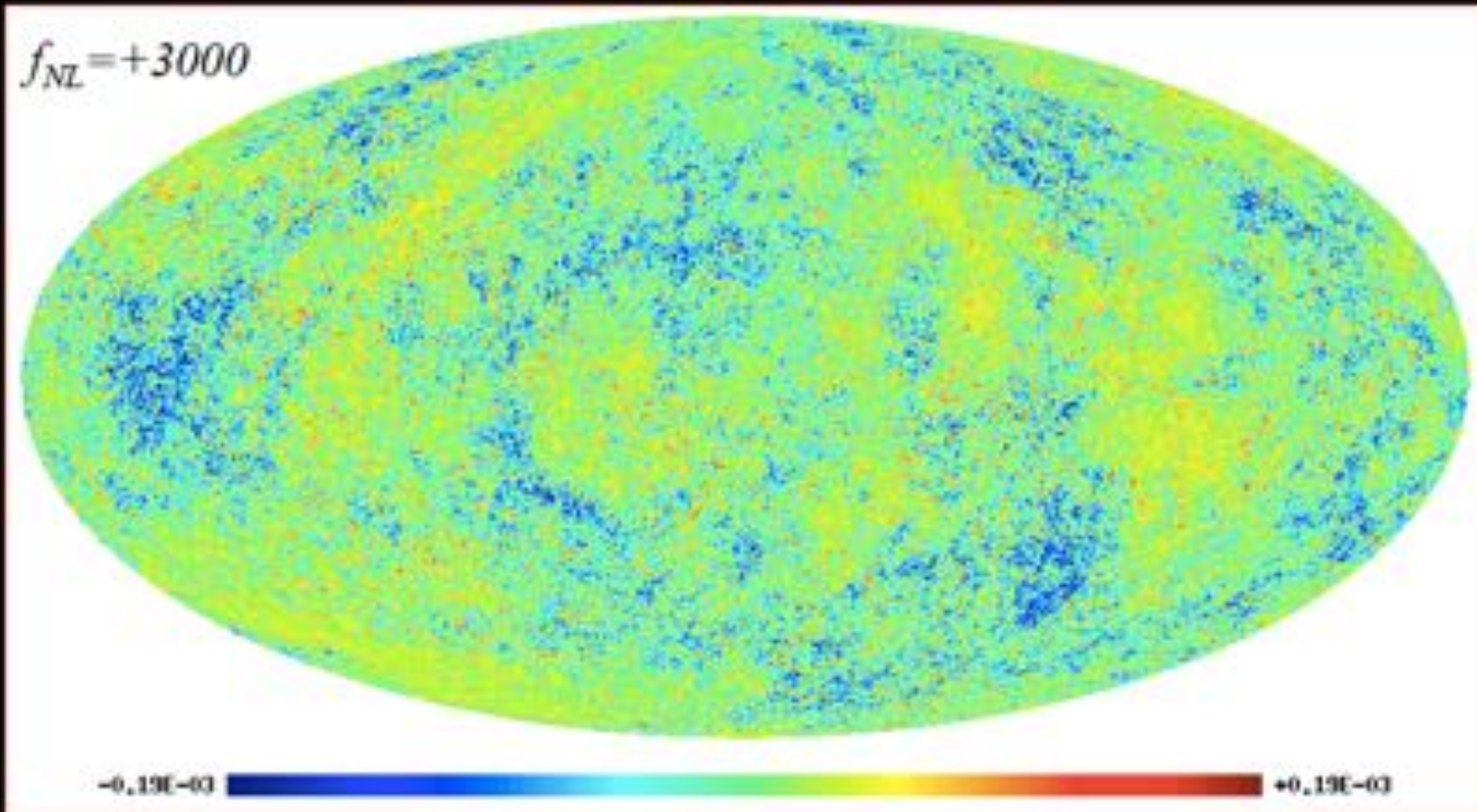


$\Delta T/T \approx -\Phi/3 \approx -\zeta/5$

Liguori, Matarrese and Moscardini (2003)

Newtonian metric ***a local function of Gaussian random field***

$$\Phi(x) = \phi_G(x) + f_{NL}\,(\,\phi_G^{\,2}(x) - \langle\phi_G^{\,2}\rangle\,)$$

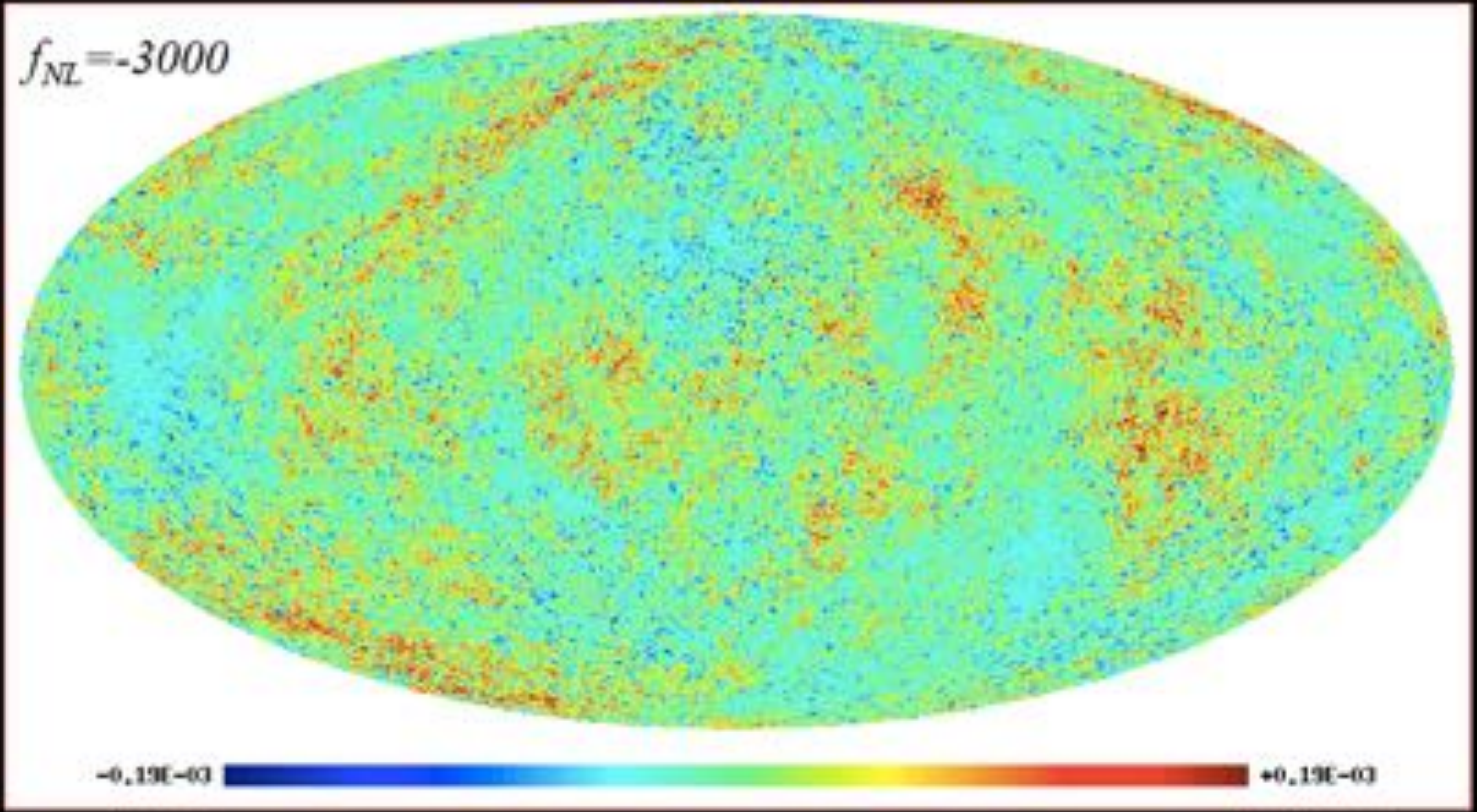


$\Delta T/T \approx -\Phi/3$, so positive $f_{NL} \Rightarrow$ more cold spots in CMB

Liguori, Matarrese and Moscardini (2003)

Newtonian potential ***a local function of Gaussian random field***

$$\Phi(x) = \phi_G(x) + f_{NL} (\phi_G^2(x) - <\phi_G^2>)$$



$\Delta T/T \approx -\Phi/3$, so negative $f_{NL} \Rightarrow$ more hot spots in CMB

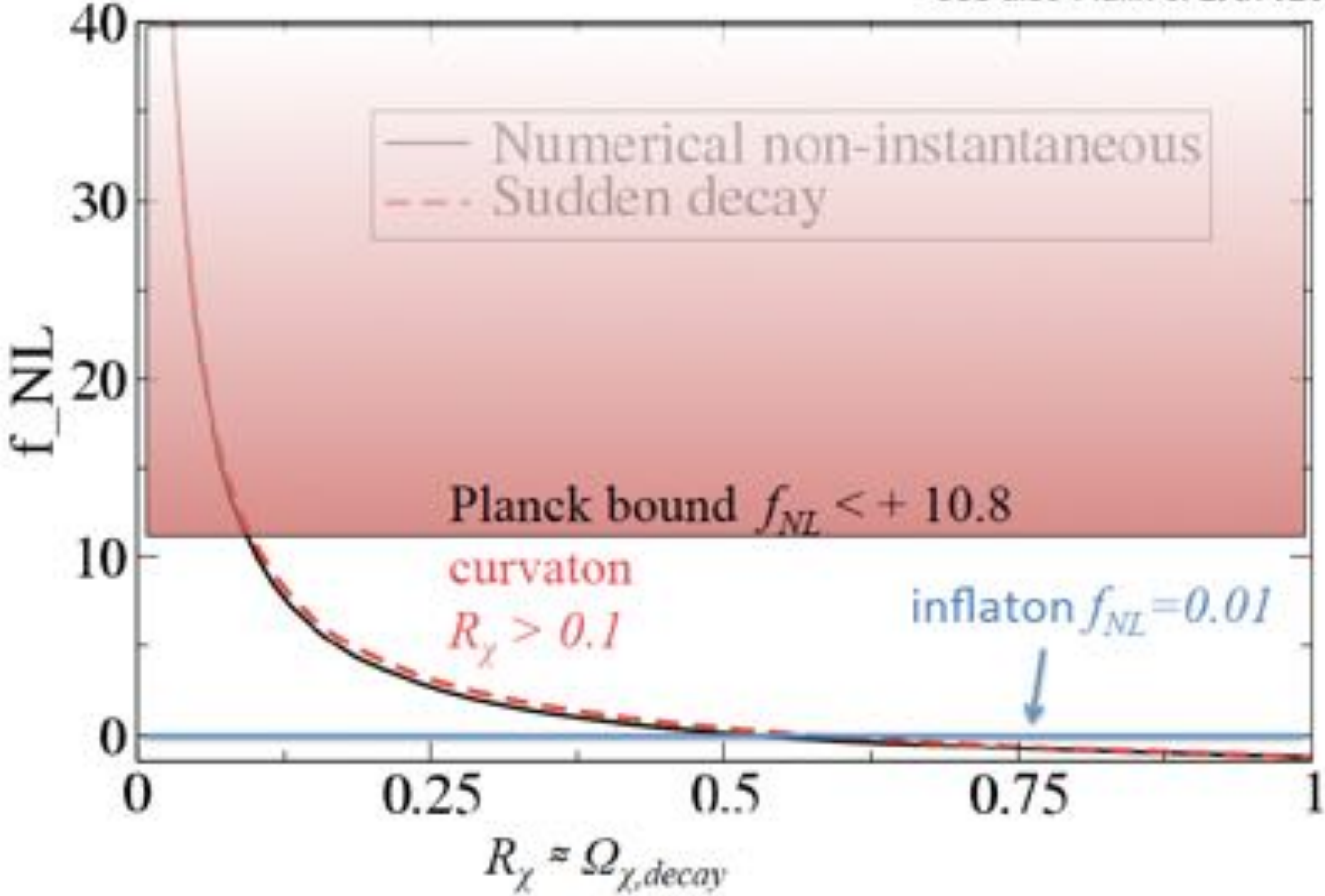
Liguori, Matarrese and Moscardini (2003)

constraints on fNL

- WMAP9 2-sigma constraints (Bennet et al 2012)
  - local: $-3 < f_{NL} < 77$
  - equilateral: $-121 < f_{NL} < 223$
  - orthogonal: $-445 < f_{NL} < -45$
- Planck2015 1-sigma constraints (Ade et al 2015)
  - local: $f_{NL} = 0.8 +/- 5.0$
  - equilateral: $f_{NL} = -4 +/- 43$
  - orthogonal: $f_{NL} = -26 +/- 21$
- SDSS BOSS galaxy power spectrum (Giannantonio et al 2015)
  - local: $f_{NL} = 5 +/- 21$

non-linearity parameter for quadratic curvaton

Sasaki, Valiviita & Wands (2006)
see also Malik & Lyth (2006)



future constraints on fNL?

- CORE-M5 1σ forecast bounds
  - local: fNL = 2.1
  - equilateral: fNL = 21
  - orthogonal: fNL = 9.6
- SDSS BOSS galaxy bispectrum 1σ forecast (Tellarini et al 2016)
  - local: fNL = 1
- BOSS + Euclid satellite 1σ forecast (Tellarini et al 2016)
  - local: fNL = 0.4

Outline

Models of inflation
Large-field inflation
Small-field inflation
Natural inflation
Higgs inflation
Starobinsky inflation

Inflationary phenomenolgy
Multi-field models and non-Gaussianity
Stochastic inflation

an inflationary landscape:

- quantum fluctuations produce stochastic noise term for coarse-grained fields on super-Hubble scales

$$\frac{d\sigma}{dN} = \frac{dV/d\sigma}{3H^2} + \frac{H}{2\pi}\xi$$

- leads to probability distribution function for local field values

$$\frac{\partial}{\partial N}P(\sigma,N) = \frac{\partial}{\partial\sigma}\left[\frac{dV/d\sigma}{3H^2}P(\sigma,N)\right] + \frac{1}{2}\left(\frac{H}{2\pi}\right)^2\frac{\partial^2}{\partial\sigma^2}P(\sigma,N)$$

stochastic inflation for the practical cosmologist

- stochastic probability distributions for initial field values in multiple field inflation:
  - the stochastic spectator (Hardwick, Vennin, Byrnes, Torrado & Wands, arXiv:1701.06473)
- stochastic delta-N (Vennin & Starobinsky, arXiv:1506.04732):
  - infinite inflation (divergent <N>) can lead to infinite perturbations
  - regularise inflation at Planck scale to get finite primordial power spectrum (Vennin et al, arXiv:1604.06017)

an inflationary cosmologist's agenda:

1. prove inflation really happened (Monday)
   - search for a smoking gun
2. show how inflation happened (Tuesday)
   - what is the correct inflation model?
3. explore inflationary phenomenology (Wednesday - Thursday)
   - primordial perturbations, gravitational waves, black holes, etc
   - particle production and reheating after inflation
4. understand the inflationary landscape (Friday)
   - multiverse, measure problem, anthropic arguments, alternatives

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end of part three