Review of Symmetries, Fields and Particles

Mathematical Tripos Part III

Easter Term, 2017

1 Lie Groups

Definition (Symmetry)**.** A *symmetry* is a transformation of dynamic variables that leaves the form of physical laws invariant.

Definition (Lie group)**.** A *Lie group* is a group manifold with dimension that of the manifold.

Remark. Smoothness reduces understanding to near the identity.

Classifying Lie groups reduces to classifying Lie algebras. Degeneracies in the spectrum of a quantum system are determined by irreducible representations of the global symmetry.

Examples.

1) $\mathrm{O}(n)$ has two disconnected pieces and is length-preserving;
2) $\mathrm{SO}(n)$ preserves the sign of the volume element $\Omega = \varepsilon_{i_1 \cdots i_n} v_1^{i_1} \cdots v_n^{i_n}$ where $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a frame in $\mathbb{R}^n$.

Examples.

1) $M(\theta) = \cos\theta \mathbb{I}_2 - \sin\theta \mathbb{J}_2 \in \mathrm{SO}(2)$, $\mathcal{M}(\mathrm{SO}(2)) = S^1$;
2) $M(\boldsymbol{\omega}) = \cos\theta \delta_{ij} + (1-\cos\theta) n_i n_j - \sin\theta \varepsilon_{ijk} n_k \in \mathrm{SO}(3)$, $\mathcal{M}(\mathrm{SO}(3)) = B_3 \cup (\partial \bar{B}_3 / \mathbb{Z}_2)$ where $\theta \equiv |\boldsymbol{\omega}|$, $\mathbf{n} \equiv \hat{\boldsymbol{\omega}}$. This is compact (closed and bounded), connected but not simply connected.

Examples. Non-compact signature-preserving group

$$\mathrm{O}(p,q) = \{M \in \mathrm{GL}(n, \mathbb{R}) : M^T \eta M = \eta\}$$

where $\eta = \left(\begin{smallmatrix} I_p & 0 \\ 0 & -I_q \end{smallmatrix}\right)$, e.g. $M = \left(\begin{smallmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{smallmatrix}\right) \in \mathrm{SO}(1,1)$.

Definition (Isomorphism)**.** $G \simeq G'$ if there exists a bijective homomorphism.

2 Lie Algebras

Definition (Lie algebra)**.** A *Lie algebra* is a vector space over a field with an antisymmetric, bilinear map known as a *Lie bracket* that satisfies the *Jacobi identity*.

Remark. A vector space V with an associative product has a natural Lie algebra. By Jaboci, the structure constants satisfy $f^{ab}{}_c f^{cd}{}_e + f^{bd}{}_c f^{ca}{}_e + f^{da}{}_c f^{cb}{}_e = 0$.

Definition (Lie algebra isomorphism)**.** $\mathfrak{g} \simeq \mathfrak{g}'$ if the underlying isomorphism preserves the Lie bracket.

Remark. Classification of Lie algebras is up to isomorphisms.

Definition (Ideal)**.** An *ideal* of $\mathfrak{g}$ is a subalgebra with strong closure, i.e. $[X, Y] \in \mathfrak{h} \, \forall \, X \in \mathfrak{h}, Y \in \mathfrak{g}$.

> *Examples.*
>
> 1) Trivial ideals $\mathfrak{h} = \{0\}, \mathfrak{g}$;
> 2) The *derived algebra* $\mathfrak{i}(\mathfrak{g}) \coloneqq [\mathfrak{g}, \mathfrak{g}] \equiv \text{span}_{\mathbb{F}}\{[X, Y] : X, Y \in \mathfrak{g}\}$;
> 3) The *centre* $\mathfrak{z}(\mathfrak{g}) \coloneqq \{X \in \mathfrak{g} : [X, Y] = 0 \, \forall \, Y \in \mathfrak{g}\}$.

Definition (Simplicity)**.** A Lie algebra $\mathfrak{g}$ is *simple* if it is non-abelian and possesses no non-trivial ideals.

Remark. For simple $\mathfrak{g}$, $\mathfrak{z}(\mathfrak{g}) = \{0\}, \mathfrak{i}(\mathfrak{g}) = \mathfrak{g}$. For abelian $\mathfrak{g}$, $\mathfrak{z}(\mathfrak{g}) = \mathfrak{g}, \mathfrak{i}(\mathfrak{g}) = \{0\}$.

3 Lie Algebras from Lie Groups

Definition (Tangent space)**.** The *tangent space* $T_p\mathcal{M}$ to $\mathcal{M}$ at p is a D-dimensional vector space spanned by $\{\partial_j\}_{j=1}^{D}$. A *tangent vector* $V = v^i \partial_i \in T_p\mathcal{M}$ acts on functions $f : \mathcal{M} \to \mathbb{R}$ as $V(f) = v^i \partial_i f(x)|_{x=0}$.

Definition (Curve)**.** A smooth curve $C : \mathbb{R} \to \mathcal{M}$ is continuous and once-differentiable.

The Lie algebra associated with a Lie group is $\mathfrak{L}(G) = (\mathcal{T}_e(G), [\cdot, \cdot])$.

> *Examples.*
>
> - $\mathfrak{L}(\text{SO}(n)) = \mathfrak{L}(\text{O}(n)) = \{\text{real skew-symmetric matrices}\}$;
> - $\mathfrak{L}(\text{SU}(n)) = \{\text{traceless skew-Hermitian matrices}\}$;
> - $\mathfrak{L}(\text{SU}(2))$ spanned by $T^a = -i\sigma_a/2$ and $\mathfrak{L}(\text{SO}(3))$ spanned by $(\tilde{T}^a)_{bc} = -\varepsilon_{abc}$ both with $f^{ab}{}_c = \varepsilon_{abc}$.
>
> *Remark.* Although $\text{SO}(3) \not\simeq \text{SU}(2)$, $\mathfrak{L}(\text{SO}(3)) = \mathfrak{L}(\text{SU}(2))$.

Definition (Translation maps)**.** The *left* and *right translations* associated with $h \in G$ are $L_h : g \mapsto hg$ and $R_h : g \mapsto gh$.

Remark. They are bijective and *diffeomorphisms* of G.

$L_h : g \mapsto hg(\theta) = g(\theta')$ is specified by $\theta' \equiv \theta'(\theta)$ with Jacobian $J^i_j = \frac{\partial \theta'^i}{\partial \theta^j}$. This induces a linear map $\forall \, g$

$$L_h^* : \mathcal{T}_g(G) \longrightarrow \mathcal{T}_{hg}(G), \quad v = v^i \frac{\partial}{\partial \theta^i} \longmapsto v' = v'^i \frac{\partial}{\partial \theta'^i},$$

where $v'^i = J^i_j(\theta) v^j$.

Definition (Left-invariant vector field)**.** The *left-invariant vector field* given $w \in \mathcal{T}_e(G)$ is $V : g \mapsto L_g^*(w)$.

Remark. This is smooth and non-vanishing.

Claim 1. $L_h^*(X) = hX \in \mathcal{T}_h(G)\ \forall\, h \in G, X \in \mathfrak{L}(G)$. *In particular,* $g^{-1}(t)\dot{g}(t) = L_{g^{-1}}^*(\dot{g}(t)) \in \mathfrak{L}(G)$.

Remark. Conversely, given $X \in \mathfrak{L}(G)$, we can construct a curve $C : \mathbb{R} \to G$ by solving the ODE $g^{-1}(t)\dot{g}(t) = X$ for all t subject to $g(0) = I_n$.

Definition (Exponential map)**.** $\text{Exp}(M) := \sum_{l=0}^{\infty} M^l/l! \in \text{Mat}_n(\mathbb{F})$ provided it converges for $M \in \text{Mat}_n(\mathbb{F})$.

Remark. The exponential map $\text{Exp} : \mathfrak{L}(G) \to G$ is bijective in some neighbourhood of e. With the correct choice of range $\mathfrak{I}$ of t, $S_{X,\mathfrak{I}} := \{g(t) = \text{Exp}(tX) : t \in \mathfrak{I} \subseteq \mathbb{R}\}$ is an abelian Lie subgroup of G.

Baker–Campbell–Hausdorff (BCH) formula.

$$\text{Exp}(X)\text{Exp}(Y) = \text{Exp}\left\{X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] - [Y,[X,Y]]\right) + \cdots\right\}.$$

Remark. Provided convergence in the BCH formula, $\mathfrak{L}(G)$ completely determines G in some neighbourhood of e. But globally the exponential map is not bijective: not surjective when G is not connected; not injective when G has a $\text{U}(1)$ subgroup.

> *Examples.*
>
> 1) $\mathfrak{L}(\text{O}(n)) = \{X \in \text{Mat}_n(\mathbb{F}) : X + X^T = 0\}$ so $\text{tr}\, X = 0$. But $\det \text{Exp}X = \exp \text{tr}\, X = 1$, $\text{Exp}(\mathfrak{L}(\text{O}(n))) = \text{SO}(n) \neq \text{O}(n)$;
> 2) $\mathfrak{L}(\text{U}(1)) = \{X = ix : x \in \mathbb{R}\}$. Since $g = \text{Exp}X = e^{ix} \in \text{U}(1)$, $ix \sim ix + 2i\pi$.

4 Representation of Lie Algebras

Definition (Representation)**.** A *representation* d of a Lie algebra is a linear homomorphism to a set of matrices preserving the Lie bracket.

Remark. $\dim d := \dim \mathcal{V} \neq \dim G$. Given representation D of a matrix Lie group G and $X \in \mathfrak{L}(G)$,

$$d(X) = \left.\frac{d}{dt}\right|_{t=0} D(g(t)).$$

> *Examples.*
>
> 1) The *trivial representation* d_0 with $d_0(X) = 0 \in \mathbb{F}$ of dimension 1;
> 2) The *fundamental representation* d_f with $d_f(X) = X$ of dimension D;
> 3) The *adjoint representation* $d_{\text{adj}}(X) = \text{ad}_X$.

Definition (Adjoint map)**.** Given $X \in \mathfrak{g}$, its *adjoint map* is $\text{ad}_X : \mathfrak{g} \to \mathfrak{g}, Y \mapsto [X,Y]$.

Remark. $\left[d_{\text{adj}}(X)\right]^b{}_c = X_a f^{ab}{}_c$ where $f^{ab}{}_c$ the structure constants of $\mathfrak{g}$.

Definition (Equivalence of representations). $R_1 \simeq R_2$ if there exists a non-singular matrix S s.t. $\forall\, X \in \mathfrak{g}$, $R_2(X) = SR_1(X)S^{-1}$.

Definition (Invariant subspace). A representation R with representation space $\mathcal{V}$ has an *invariant subspace* $\mathcal{U} \subseteq \mathcal{V}$ if $R \cdot \mathcal{U} \subseteq \mathcal{U}$.

Remark. $\mathcal{U} = \{0\}, \mathcal{V}$ are trivial invariant subspaces.

Definition (Irreducibility). An *irreducible representation* (irrep) of a Lie algebra has no non-trivial invariant subspaces.

Representations of $\mathfrak{L}$(SU(2))

Roots. In basis $H = \sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), E_\pm = (\sigma_1 \pm i\sigma_2)/2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, the *roots* of $\mathfrak{L}$(SU(2)) are the eigenvalues $\{0, \pm 2\}$ of eigenvectors $\{H, E_\pm\}$ of ad_H.

Weights. Given representation R that $R(H)$ is diagonalisable, its eigenvectors span $\mathcal{V}$ and its eigenvalues $\{\lambda\}$ are known as the *weights* of representation R.

Step operators. $E_\pm$ obey $R(H)R(E_\pm)v_\lambda = (\lambda \pm 2)R(E_\pm)v_\lambda$.

Results. For a finite-dimensional, irreducible representation R_Λ of $\mathfrak{L}$(SU(2)) labelled by the highest weight $\Lambda \in \mathbb{N}$,

1) the weight set is $S_R = \{-\Lambda, -\Lambda + 2, \ldots, \Lambda - 2, \Lambda\} \subset \mathbb{Z}$;
2) the weights are non-degenerate with $\dim(R_\Lambda) = \Lambda + 1$.

Representations from $\mathfrak{L}$(SU(2))

SU(2) **representations.** Obtained from Exp : $R_\Lambda(X) \mapsto D_\Lambda(A)$.

SO(3) **versus SU**(2). SO(3) = SU(2)/$\mathbb{Z}_2$ requires $D_\Lambda(I_2) = D_\Lambda(-I_2)$, but

$$-I_2 = \mathrm{Exp}(i\pi H), \quad H = \mathrm{diag}(1, -1)$$

so $D_\Lambda(-I_2) = \mathrm{Exp}(i\pi R_\Lambda(H))$ has eigenvalues $e^{i\pi\lambda} = (-1)^\lambda = (-1)^\Lambda$:

1) $\Lambda \in 2\mathbb{Z}$, then D_Λ represents both SU(2) and SO(3);
2) $\Lambda \in 2\mathbb{Z} + 1$, then D_Λ represents SU(2) but not SO(3).

5 Representation Theory

Definition (Conjugate representation). The *conjugate representation* of a representation R of a real Lie algebra $\mathfrak{g}$ is $\bar{R}(X) = R(X)^* \,\forall\, X \in \mathfrak{g}$.

Remark. Possibly $\bar{R} \simeq R$.

Direct sum. The direct sum $R_1 \oplus R_2$ is a representation acting on $V_1 \oplus V_2 = \{v_1 \oplus v_2\}$,

$$(R_1 \oplus R_2)(X)(v_1 \oplus v_2) = R_1(X)v_1 \oplus R_2(X)v_2$$

with the matrix $(R_1 \oplus R_2)(X) = \left(\begin{smallmatrix} R_1(X) & 0 \\ 0 & R_2(X) \end{smallmatrix}\right)$ and $\dim(R_1 \oplus R_2) = \dim R_1 + \dim R_2$.

Tensor product. The tensor product $R_1 \otimes R_2$ is a representation acting on $V_1 \otimes V_2 = \{v_1 \oplus v_2\}$,

$$(R_1 \otimes R_2)(X) = R_1(X) \otimes I_{(2)} + I_{(1)} \otimes R_2(X)$$

with the matrix $(R_1 \otimes R_2)(X)_{i\alpha,j\beta} = R_1(X)_{ij} I_{\alpha\beta} + I_{ij} R_2(X)_{\alpha\beta}$ and $\dim(R_1 \otimes R_2) = \dim R_1 \dim R_2$.

Remark. If R is reducible, there is a basis in which $R(X) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \forall X \in \mathfrak{g}$. If R is *fully reducible*, there exists a basis in which $R(X) = \bigoplus_i R_i(X) \forall X \in \mathfrak{g}$ for irreps R_i.

Fact 1. If R_i are finite-dimensional irreducible representations of a simple Lie algebra, then $\bigotimes_{i=1}^m R_i = \bigoplus_{j=1}^{\tilde{m}} \tilde{R}_j$ is fully reducible into irrep $\tilde{R}_j$.

> *Examples.* Let $R_\Lambda, R_{\Lambda'}$ be irreducible representations of $\mathfrak{L}(\mathrm{SU}(2))$ then
>
> $$R_\Lambda \otimes R_{\Lambda'} = \bigoplus_{\Lambda'' \in \mathbb{N}} l_{\Lambda,\Lambda'}^{\Lambda''} R_{\Lambda''}$$
>
> where $l_{\Lambda,\Lambda'}^{\Lambda''} \in \mathbb{N}$ are the *Littlewood–Richardson coefficients.* Note $S_{\Lambda,\Lambda'} = \{\lambda + \lambda' : \lambda \in S_\Lambda, \lambda' \in S_{\Lambda'}\}$ and $l_{\Lambda,\Lambda'}^{\Lambda+\Lambda'} = 1$. Example: $R_1 \otimes R_1 = R_0 \oplus R_2$ and $l_{1,1}^{\Lambda''} = \delta_{\Lambda'',2} + \delta_{\Lambda'',0}$.

Definition (Inner product)**.** An *inner product* is a symmetric bilinear form $V \times V \to \mathbb{F}$. It is *non-degenerate* if $\forall\, v \in V \backslash \{0\}, \exists\, w \in V$ s.t. $(v, w) \neq 0$.

Definition (Killing form)**.** The *Killing form* is

$$\begin{aligned} \kappa : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathbb{F} \\ (X, Y) &\longmapsto \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y). \end{aligned}$$

Remark. $\kappa^{ab} = f^{ad}{}_c f^{bc}{}_d$.

Invariance under adjoint action. $\kappa(X, [Y, Z]) + \kappa(Y, [X, Z]) = 0$.

Fact 2. If $\mathfrak{g}$ is simple, the Killing form κ gives rise to the unique inner product (up to constant rescaling) that is invariant under the transformation $\delta_Z : X \mapsto X + [Z, X]$.

Definition (Semi-simplicity)**.** A Lie algebra is *semi-simple* if it has no non-zero abelian ideals.

Theorem 2. *If $\mathfrak{g}$ is finite-dimensional and semi-simple, it is the direct sum of finitely many simple Lie algebras.*

Theorem 3 (Cartan)**.** *The Killing form κ is non-degenerate iff the Lie algebra $\mathfrak{g}$ is semi-simple.*

Remark. Complex Lie algebras may have more than one real form, e.g. both $\mathfrak{L}(\mathrm{SU}(2))$ and $\mathfrak{L}(\mathrm{SL}(2, \mathbb{R}))$ are complexified to $\mathfrak{L}_{\mathbb{C}}(\mathrm{SU}(2))$.

> *Examples.* $\mathfrak{L}(\mathrm{SU}(2)) = \{2 \times 2 \text{ traceless skew-Hermitian matrices}\}$,
> $\mathfrak{L}_{\mathbb{C}}(\mathrm{SU}(2)) = \{2 \times 2 \text{ traceless complex matrices}\} \simeq \mathfrak{L}(\mathrm{SL}(2,\mathbb{C}))$.

Definition (Compact type)**.** A real Lie algebra is of *compact type* if there is a basis s.t. $\kappa^{ab} = -\kappa\delta^{ab}, \kappa > 0$.

Theorem 4. *Every finite-dimensional complex semi-simple Lie algebra has a real form of compact type.*

6 Cartan Classification of Finite-Dimensional Simple Complex Lie Algebras

Definition (Adjointly diagonalisable)**.** $X \in \mathfrak{g}$ is *adjointly diagonalisable* (a.d.) if $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is diagonalisable.

Definition (Cartan subalgebra)**.** A *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal abelian subalgebra containing only a.d. elements.

Fact 3. All possible Cartan subalgebras $\mathfrak{h} \subset \mathfrak{g}$ have the same dimension $r \equiv \dim \mathfrak{h}$ known as the *rank* of $\mathfrak{g}$.

> *Examples.* For $\mathfrak{g} = \mathfrak{L}_{\mathbb{C}}(\mathrm{SU}(n))$ consisting of traceless complex matrices, $(H^i)_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta i} - \delta_{\alpha i+1}\delta_{\beta i+1}, 1 \leqslant i \leqslant n-1$. Hence $\mathrm{rank}\,\mathfrak{g} = n-1$.

Properties.

1) $H \in \mathfrak{h}$ implies H is a.d.;
2) $H, H' \in \mathfrak{h} \Rightarrow [H, H'] = 0 \Rightarrow \mathrm{ad}_H \circ \mathrm{ad}_{H'} = \mathrm{ad}_{H'} \circ \mathrm{ad}_H$;
3) $X \in \mathfrak{g}$ and $[X, H] = 0\,\forall\, H \in \mathfrak{h}$ imply $X \in \mathfrak{h}$.

Remark. $[H^i, H^j] = 0$ so ad_{H^i} are simultaneously diagonalisable. The spectrum includes:

1) zero eigenvalues $\{H^j : j = 1, \ldots, r\}$;
2) nonzero eigenvalues $\{E^\alpha : \alpha \in \Phi\}$ for which $\mathrm{ad}_{H^i}(E^\alpha) = \alpha^i E^\alpha$, where α are *roots*.

Fact 4. *Roots* $\alpha : \mathfrak{h} \to \mathbb{C}$ of $\mathfrak{g}$ are non-degenerate elements of the dual vector space $\mathfrak{h}^*$.

Remark. $\alpha : H = e_i H^i \mapsto \alpha^i e_i$ since $[H, E^\alpha] = \alpha(H) = \alpha^i e_i E^\alpha$.

Definition (Cartan–Weyl basis)**.** The *Cartan–Weyl basis* for $\mathfrak{g}$ is

$$\mathcal{B} = \{H^i : i = 1, \ldots, r\} \cup \{E^\alpha : \alpha \in \Phi\}$$

satisfying $[H^i, H^j] = 0, [H^i, E^\alpha] = \alpha^i E^\alpha$.

Remark. $|\Phi| = \dim \mathfrak{g} - \mathrm{rank}\,\mathfrak{g}$.

Definition (Killing form)**.** On the simple Lie algebra $\mathfrak{g}$

$$\kappa(X, Y) = \frac{1}{N}\,\mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$$

for some normalisation constant $N > 0$.

Remark. By simplicity, κ is non-degenerate by Cartan's theorem.

Proposition 5.

1) $\kappa(H, E^\alpha) = 0 \,\forall\, H \in \mathfrak{h}, \alpha \in \Phi$;
2) $\kappa(E^\alpha, E^\beta) = 0 \,\forall\, \alpha, \beta \in \Phi : \alpha + \beta \neq 0$;
3) $\forall\, H \in \mathfrak{h}, \exists\, H' \in \mathfrak{h}$ *s.t.* $\kappa(H, H') \neq 0$;
4) $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$ *and* $\kappa(E^\alpha, E^{-\alpha}) \neq 0$.

Remark. (3) says κ is non-degenerate on $\mathfrak{h}$, inducing a non-degenerate inner product on $\mathfrak{h}^*$

$$(\alpha, \beta) = (\kappa^{-1})_{ij} \alpha^i \beta^j,$$

and an isomorphism $K : H \in \mathfrak{h} \mapsto \kappa(H, \cdot\,) \in \mathfrak{h}^*$.

Result. By invariance of the Killing form,

$$[H^i, [E^\alpha, E^\beta]] = (\alpha^i + \beta^i)[E^\alpha, E^\beta]$$
$$\kappa([E^\alpha, E^{-\alpha}], H) = \alpha(H)\kappa(E^\alpha, E^{-\alpha}) \neq 0$$

so $\kappa(H^\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}$ has the unique solution

$$H^\alpha = \frac{[E^\alpha, E^{-\alpha}]}{\kappa(E^\alpha, E^{-\alpha})}$$

by non-degeneracy, i.e. $H^\alpha = (\kappa^{-1})_{ij} \alpha^j H^i$.

Cartan-Weyl algebra.

$$e^\alpha = \sqrt{\frac{2}{(\alpha, \alpha)\kappa(E^\alpha, E^{-\alpha})}} E^\alpha, \quad h^\alpha = \frac{2}{(\alpha, \alpha)} H^\alpha$$

satisfies

$$[h^\alpha, h^\beta] = 0, \quad [h^\alpha, e^\beta] = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} e^\beta \tag{1}$$

$$[e^\alpha, e^\beta] = \begin{cases} n_{\alpha,\beta} e^{\alpha+\beta}, & \alpha + \beta \in \Phi \\ h^\alpha, & \alpha + \beta = 0 \\ 0, & \text{else.} \end{cases} \tag{2}$$

$\mathfrak{sl}(2)_\alpha$ subalgebra. $[h^\alpha, e^{\pm\alpha}] = \pm 2 e^{\pm\alpha}, [e^\alpha, e^{-\alpha}] = h^\alpha$.

Definition (Root string)**.** For roots $\beta \not\propto \alpha$ in Φ, the *α-string passing through β* is

$$S_{\alpha,\beta} = \{\beta + n\alpha \in \Phi : n \in \mathbb{Z}\}.$$

Remark. The corresponding vector subspace

$$V_{\alpha,\beta} = \text{span}_{\mathbb{C}}\{e^{\beta+n\alpha} \in \mathfrak{g} : n \in \mathbb{Z}\}$$

is an invariant subspace under $\mathfrak{sl}(2)_\alpha$, thus is the representation space for some representation R of $\mathfrak{sl}(2)_\alpha$, with weight set

$$S_R = \left\{ 2\left[n + \frac{(\alpha, \beta)}{(\alpha, \alpha)} \right] : \beta + n\alpha \in \Phi, n_- \leqslant n \leqslant n_+, n \in \mathbb{Z} \right\}, \quad \frac{2(\alpha, \beta)}{(\alpha, \alpha)} = -(n_+ + n_-).$$

Proposition 6. $(\alpha, \beta) \in \mathbb{R}$.

Lemma 7. $\mathfrak{h}^* = \operatorname{span}_{\mathbb{C}}\{\alpha : \alpha \in \Phi\}$.

Corollary 8. $\dim \mathfrak{g} \geqslant 2 \operatorname{rank} \mathfrak{g}$.

Lemma 9. $\Phi \subset \mathfrak{h}^*_{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\{\alpha_{(i)} \in \Phi : i = 1, \cdots, r\}$.

Proposition 10. *Roots $\alpha \in \Phi$ are elements of the real vector space $\mathfrak{h}^*_{\mathbb{R}} \simeq \mathbb{R}^r$ where $r = \operatorname{rank} \mathfrak{g}$, equipped with a Euclidean inner product $(\,\cdot\,,\,\cdot\,)$ s.t. for all $\lambda, \mu \in \mathfrak{h}^*_{\mathbb{R}}$,*

1) $(\lambda, \mu) \in \mathbb{R}$;
2) $(\lambda, \lambda) \geqslant 0$ *with equality iff* $\lambda = 0$.

Definition (Norm and angle)**.** The norm of a root α is

$$|\alpha| := \sqrt{(\alpha, \alpha)} > 0.$$

The angle between any two roots, $\phi \equiv \measuredangle(\alpha, \beta)$, is given by

$$(\alpha, \beta) = |\alpha||\beta| \cos\phi, \quad \phi \in [0, \pi].$$

Lemma 11. $4\cos^2\phi \in \{0, 1, 2, 3, 4\}$.

Definition (Simple root)**.** A *simple root* $\delta \in \Phi_S$ is a positive root that cannot be written as a sum of two positive roots.

Proposition 12.

1) *If $\alpha, \beta \in \Phi_S$, then $\alpha - \beta$ is not a root;*
2) *If $\alpha, \beta \in \Phi_S$, then the length of the α-string passing through β is*

$$l_{\alpha,\beta} = 1 - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{N}\backslash\{0\};$$

3) *If $\alpha, \beta \in \Phi_S$ and $\alpha \neq \beta$, $(\alpha, \beta) \leqslant 0$;*
4) *Any positive root can be written as a linear combination of simple roots with positive integer coefficients, i.e.*

$$\beta \in \Phi_+ \quad \Longrightarrow \quad \beta = \sum_i c_i \alpha_{(i)},\ \alpha_{(i)} \in \Phi_S,\ c_i \in \mathbb{N};$$

5) *Simple roots are linearly independent;*
6) *There are exactly $r = \operatorname{rank} \mathfrak{g}$ simple roots, i.e. $|\Phi_S| = r$.*

Definition. Let $\mathcal{B} = \{\alpha_{(i)} \in \Phi_S : i = 1, \ldots, r\}$ be an enumerated basis for $\mathfrak{h}^*_{\mathbb{R}}$. The *Cartan matrix* A is

$$A^{ij} := 2\frac{(\alpha_{(i)}, \alpha_{(j)})}{(\alpha_{(j)}, \alpha_{(j)})} \in \mathbb{Z}, \quad i, j = 1, \ldots, r.$$

Simple root algebra. For each $\alpha_{(i)} \in \Phi_S$ there is an associated $\mathfrak{sl}(2) = \mathrm{span}\{h^i \equiv h^{\alpha_{(i)}}, e^i_\pm \equiv e^{\pm\alpha_{(i)}}\}$ obeying

$$[h^i, e^i_\pm] = \pm 2 e^i_\pm, \quad [e^i_+, e^i_-] = h^i.$$

The 'Cartan–Weyl algebra' becomes

$$\begin{aligned} [h^i, h^j] &= 0 \\ [h^i, e^j_\pm] &= \pm A^{ji} e^j_\pm \\ [e^i_+, e^j_-] &= \delta^{ij} h^i. \end{aligned}$$

(Chevalley–)Serra relation. $\mathrm{ad}^{1-A^{ji}}_{e^i_\pm} e^j_\pm = 0$.

Theorem 13 (Cartan)**.** *A finite-dimensional simple complex Lie algebra is uniquely determined by its Cartan matrix.*

Remark. The Cartan matrix determines simple roots $\alpha_{(i)}, i = 1, \ldots, r$ up to the choice of the first vector $\alpha_{(1)} \in \mathbb{R}^r$, and the remaining via root strings

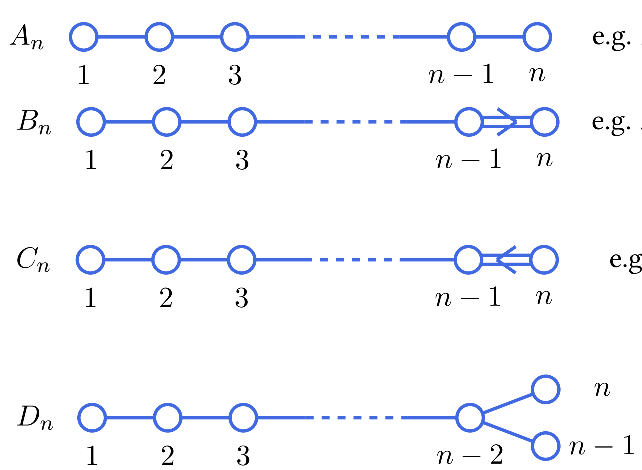
Constraints.

1) $A^{ii} = 2,\ i = 1, \ldots, r;$
2) $A^{ij} = 0 \ \Leftrightarrow\ A^{ji} = 0;$
3) $A^{ij} \in \mathbb{Z}_{\leqslant 0}$ for $i \neq j$ by property 3) of simple roots;
4) $\det A > 0$ by non-degeneracy of the Euclidean inner product $(\,\cdot\,,\,\cdot\,)$;
5) A is irreducible.

Remark. $\dfrac{|\alpha_{(i)}|}{|\alpha_{(j)}|} = \sqrt{\dfrac{A^{ij}}{A^{ji}}}, \quad \cos^2 \phi_{ij} = \dfrac{1}{4} A^{ij} A^{ji}.$

Lemma 14. *A simple Lie algebra has simple roots of at most two different lengths.*

Cartan classification.

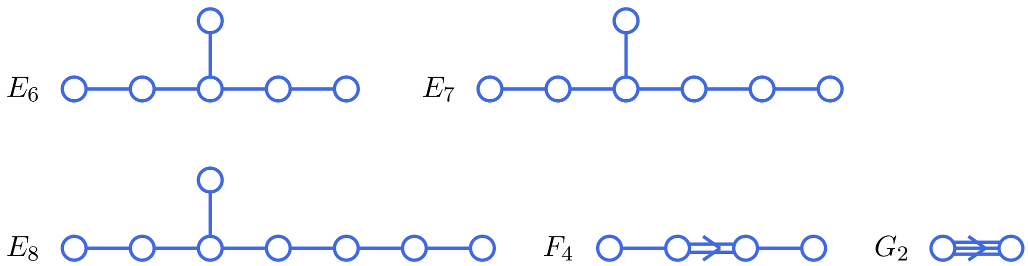


A_n: e.g. $\mathfrak{L}_\mathbb{C}(\mathrm{SU}(n+1))$

B_n: e.g. $\mathfrak{L}_\mathbb{C}(\mathrm{SO}(2n+1))$

C_n: e.g. $\mathfrak{L}_\mathbb{C}(\mathrm{Sp}(2n))$

D_n: e.g. $\mathfrak{L}_\mathbb{C}(\mathrm{SO}(2n))$



Remark.

1) $n = 1$, $A_1 = B_1 = C_1 = D_1 \simeq \mathfrak{L}_{\mathbb{C}}(\mathrm{SU}(2))$, e.g. $\mathfrak{L}_{\mathbb{C}}(\mathrm{SU}(2)) \simeq \mathfrak{L}_{\mathbb{C}}(\mathrm{SO}(3))$;
2) $n = 2$, $B_2 = C_2$ and $D_2 \simeq A_1 \oplus A_1$;
3) $n = 3$, $D_3 = A_3$.

Representation of simple Lie algebras. Consider a representation R of the simple Lie algebra $\mathfrak{g}$ acting on representation space $R(H^i)R(E^\alpha)v = (\lambda^i + \alpha^i)R(E^\alpha)v$, i.e. each weight λ is shifted by roots α under the action of step operators.

Remark. $R(h^\alpha)v_\lambda = \dfrac{2(\alpha, \lambda)}{(\alpha, \alpha)}v_\lambda$ so $\dfrac{2(\alpha, \lambda)}{(\alpha, \alpha)} \in S_{R_\alpha}$ for some representation R_α of $\mathfrak{sl}(2)$.

Definition (Co-root and lattices)**.** *Simple co-roots* $\alpha^\vee_{(i)} = \dfrac{2\alpha_{(i)}}{(\alpha_{(i)}, \alpha_{(i)})}$. The *root lattice* and *co-root* lattice are

$$L[\mathfrak{g}] \coloneqq \mathrm{span}_{\mathbb{Z}}\{\alpha_{(i)} : i = 1, \ldots, r\}, \quad L^\vee[\mathfrak{g}] \coloneqq \mathrm{span}_{\mathbb{Z}}\{\alpha^\vee_{(i)} : i = 1, \ldots, r\}.$$

The *weight lattice* is dual to the co-root lattice

$$L_W[\mathfrak{g}] \coloneqq L^{\vee *}[\mathfrak{g}] \equiv \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} : (\lambda, \mu) \in \mathbb{Z} \, \forall \, \mu \in L^\vee[\mathfrak{g}]\}.$$

Remark. All weights are in the weight lattice $S_R \subset L_W[\mathfrak{g}]$.

Definition. Given a basis $\mathcal{B} = \{\alpha^\vee_{(i)} : i = 1, \ldots, r\}$ of the co-root lattice $L^\vee[\mathfrak{g}]$, the *fundamental weights* of $\mathfrak{g}$ are the *dual basis* $\mathcal{B}^* = \{\omega_{(i)} : i = 1, \ldots, r\}$ for $L_W[\mathfrak{g}]$ satisfying $(\alpha^\vee_{(i)}, \omega_{(j)}) = \delta_{ij}$.

Remark. $\alpha_{(i)} = \sum_{j=1}^r A^{ij}\omega_{(j)}$.

Definition (Dynkin labels)**.** For any weight $\lambda \in S_R \subseteq L_W[\mathfrak{g}]$, $\lambda = \sum_{i=1}^r \lambda^i \omega_{(i)}$ where $\{\lambda^i\}$ are the *Dynkin labels* of λ.

Definition (Highest weight)**.** The *highest weight* Λ of a representation R has its eigenvector $v_\Lambda \in V$ annihilated by all step operators

$$R(E^\alpha)v_\Lambda = 0 \qquad \forall \, \alpha \in \Phi_+.$$

Definition (Dynkin labels)**.** Given any finite-dimensional representation R of $\mathfrak{g}$ labelled by its highest weight $\Lambda = \sum_{i=1}^r \Lambda^i \omega_{(i)} \in S_R$, its *Dynkin labels* are $\{\Lambda^i \in \mathbb{Z}\}$.

Fact 5. For any finite-dimensional representation R of $\mathfrak{g}$,

$$\lambda = \sum_{i=1}^r \lambda^i \omega_{(i)} \in S_R \quad \Longrightarrow \quad \lambda - m_{(i)}\alpha_{(i)} \in S_R$$

where $0 \leqslant m_{(i)} \leqslant \lambda^i, m_{(i)} \in \mathbb{N}$.

Definition (Dominant integral weight)**.** $\lambda = \sum_i \lambda^i \omega_{(i)}$ is a *dominant integral weight* if $\lambda^i \in \mathbb{N}$. Denote the set of dominant integral weights by $\overline{L}_W$.

Irreducible Representations of A_2

Fact 6. Each dominant integral weight in A_2

$$\Lambda = \Lambda^1\omega_{(1)} + \Lambda^2\omega_{(2)} \in \overline{L}_W, \qquad \Lambda^{1,2} \in \mathbb{N}$$

gives an irreducible representation (irrep.) $R_{(\Lambda^1,\Lambda^2)}$ of dimension

$$\dim R_{(\Lambda^1,\Lambda^2)} = \frac{1}{2}(\Lambda^1+1)(\Lambda^2+1)(\Lambda^1+\Lambda^2+2).$$

For $\Lambda^1 \neq \Lambda^2$, $R_{(\Lambda^2,\Lambda^1)} = \overline{R}_{(\Lambda^1,\Lambda^2)}$ with their weights related by reflection: $\lambda \in S_{(\Lambda^1,\Lambda^2)} \Leftrightarrow -\lambda \in S_{(\Lambda^2,\Lambda^1)}$.

Claim 15. $\lambda \in S_\Lambda$, $\lambda' \in S_{\Lambda'} \Rightarrow \lambda+\lambda' \in L_W[\mathfrak{g}]$ *and* $\lambda+\lambda' \in S_{R_\Lambda \otimes R_{\Lambda'}}$.

Table 1: A_2 irreps of lowest dimensions.

Repn.	Notn.	Remarks
$R_{(0,0)}$	$\mathbf{1}$	trivial
$R_{(1,0)}$	$\mathbf{3}$	fundamental
$R_{(0,1)}$	$\overline{\mathbf{3}}$	anti-fundamental
$R_{(1,1)}$	$\mathbf{8}$	adjoint

Conclusion. Let R_Λ, labelled by the highest weight $\Lambda \in \overline{L}_W[\mathfrak{g}]$, represent irreducibly the finite-dimensional, simple, complex Lie algebra $\mathfrak{g}$:

1) Every such $\mathfrak{g}$ has a real form of compact type with $\kappa^{ab} = -\kappa\delta^{ab}, \kappa > 0$;
2) $\mathfrak{g}_\mathbb{R} = \mathfrak{L}(G)$ is classified by Cartan;
3) Every irrep R_Λ of $\mathfrak{g}$ provides an irrep R_Λ of $\mathfrak{g}_\mathbb{R}$ as well as an irrep $D_\Lambda = \mathrm{Exp}(R_\Lambda)$ of G. Further, D_Λ is unitary so $R_\Lambda(X)^\dagger + R_\Lambda(X) = 0$ for all $X \in \mathfrak{g}_\mathbb{R}$.

7 Gauge Theory

Definition. In relativistic electromagnetism, the 4-potential is $a_\mu := (\Phi, \mathbf{A})$ with the *field strength tensor* $f_{\mu\nu} := \partial_\mu a_\nu - \partial_\nu a_\mu$.

Remark. Under the gauge transformation $a_\mu \to a_\mu + \partial_\mu\chi$. Re-define $A_\mu = -ia_\mu \in i\mathbb{R} \simeq \mathfrak{L}(\mathrm{U}(1))$ and $F_{\mu\nu} = -if_{\mu\nu}$.

Definition (Global U(1)-gauge scalar field)**.** A *global* U(1)-gauge complex scalar field $\phi : \mathbb{R}^{3,1} \to \mathbb{C}$ with Lagrangian density

$$\mathcal{L}_\phi = \partial_\mu\phi^*\partial^\mu\phi - W(\phi^*\phi)$$

is invariant under U(1) global symmetry $\phi \to g\phi$, where $g = e^{i\delta} \in \mathrm{U}(1)$.

[To couple the scalar field to EM and obtain a quantum theory describing scalar 'electrons' interacting with photons, we gauge the U(1) symmetry.]

Definition (Local U(1)-gauge scalar field)**.** Promoting the above to be $g : \mathbb{R}^{3,1} \to \mathrm{U}(1)$ and $X : \mathbb{R}^{3,1} \to \mathfrak{L}(\mathrm{U}(1))$, we obtain a *local* U(1)-gauge complex scalar field $\phi : \mathbb{R}^{3,1} \to \mathbb{C}$ with Lagrangian density

$$\mathcal{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + (\mathrm{D}_\mu\phi)^*\mathrm{D}^\mu\phi - W(\phi^*\phi),$$

invariant under U(1) local symmetry

$$\delta_X \phi = \epsilon X \phi, \quad \delta_X A_\mu = -\epsilon \partial_\mu X,$$

i.e. $a_\mu \to a_\mu + \partial_\mu \chi$ with $\chi = -i\epsilon X$, where the U(1) gauge field $A_\mu : \mathbb{R}^{3,1} \to \mathfrak{L}(\mathrm{U}(1)) \simeq i\mathbb{R}$ and the *covariant derivative* $\mathrm{D}_\mu \coloneqq \partial_\mu + A_\mu$.

> *Exercise.* Show the kinetic term $(\mathrm{D}_\mu \phi)^* \mathrm{D}^\mu \phi$ is invariant under gauge transformations from $\delta_X(\mathrm{D}_\mu \phi) = \epsilon X \mathrm{D}_\mu \phi$.

Definition (Global gauge scalar field)**.** Let G be a gauge Lie group with unitary representation D, i.e. $D_\Lambda(g)^\dagger D_\Lambda(g) = \mathbb{I} \,\forall\, g \in G$, and a representation space $\mathcal{V} \simeq \mathbb{C}^N$ equipped with the standard inner product $(u, v) = u^\dagger \cdot v, u, v \in \mathcal{V}$. A *global* gauge scalar field $\phi : \mathbb{R}^{3,1} \to \mathcal{V}$ has a Lagrangian

$$\mathcal{L}_\phi = (\partial_\mu \phi, \partial^\mu \phi) - W((\phi, \phi))$$

invariant under the global symmetry transformation $\phi \to D(g)\phi \,\forall\, g \in G$.

Remark. Near the identity $g = \mathrm{Exp}(\epsilon X)$ and $D(g) = \mathrm{Exp}(\epsilon R(X))$ where $R : \mathfrak{L}(G) \to \mathrm{Mat}_N(\mathbb{C})$ is the representation of the Lie algebra satisfying $R(X)^\dagger + R(X) = 0 \,\forall\, X \in \mathfrak{L}(G)$. Infinitesimally, $D(g) \simeq \mathbb{I} + \epsilon R(X)$ and

$$\phi \longrightarrow \phi + \delta_X \phi, \quad \delta_X \phi = \epsilon R(X) \phi \in \mathcal{V}.$$

Definition (Local gauge scalar field)**.** Promoting the above to $X : \mathbb{R}^{3,1} \to \mathfrak{L}(G)$, we obtain a *local* gauge scalar field ϕ with the gauge-invariant Lagrangian

$$\mathcal{L} = (\mathrm{D}_\mu \phi, \mathrm{D}^\mu \phi) - W((\phi, \phi)),$$

the gauge field $A_\mu : \mathbb{R}^{3,1} \to \mathfrak{L}(G)$ and transformations

$$\delta_X \phi = \epsilon R(X(x)) \phi \in \mathcal{V}, \quad \delta_X A_\mu = -\epsilon \partial_\mu X + \epsilon [X, A_\mu] \in \mathfrak{L}(G).$$

where the covariant derivative $\mathrm{D}_\mu \coloneqq \partial_\mu + R(A_\mu)$.

> *Exercise.* Show the kinetic term $(\mathrm{D}_\mu \phi)^* \mathrm{D}^\mu \phi$ is invariant under gauge transformations from $\delta_X(\mathrm{D}_\mu \phi) = \epsilon R(X) \mathrm{D}_\mu \phi$.

Definition (Field strength tensor)**.** The *field strength tensor* $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \in \mathfrak{L}(G)$.

Remark. The first two terms are linear and the bracket is quadratic in A_μ, allowing us to rescale so that the coefficient of the bracket is 1. Hence this definition is general.

Claim 16. $\delta_X(F_{\mu\nu}) = \epsilon [X, F_{\mu\nu}] \in \mathfrak{L}(G)$.

Definition (Yang–Mills Lagrangian)**.** $\mathcal{L}_A = \dfrac{1}{g^2} \kappa(F_{\mu\nu}, F^{\mu\nu})$.

Remark. This is gauge-invariant $\delta_X \mathcal{L}_A = 0$ due to the invariance property of the Killing form.

Construction of gauge-invariant theories. By simplicity, the Lie algebra associated with the gauge symmetry has a real form of compact type, providing a sensible kinetic term for the gauge field. In other words, there is a basis $\mathcal{B} = \{T^a\}_{a=1}^{d \equiv \dim G}$ s.t. $\kappa^{ab} \equiv \kappa(T^a, T^b) = -\kappa \delta^{ab}, \kappa > 0$. Hence with $F_{\mu\nu} = (F_{\mu\nu})_a T^a \in \mathfrak{L}(G)$

$$\mathcal{L}_A = -\frac{\kappa}{g^2} \sum_{a=1}^{d} (F_{\mu\nu})_a (F^{\mu\nu})^a.$$

A large family of consistent theories are provided by the Cartan Classification with data:

1) gauge group

$$G \text{ (compact, simple)} \quad \rightarrow \quad \mathfrak{g}_{\mathbb{R}} = \mathfrak{L}(G) \text{ (of compact type, simple)}$$

with associated gauge field $A_\mu : \mathbb{R}^{3,1} \to \mathfrak{L}(G)$ satisfying its transformation rule;

2) matter content

$$\phi_\Lambda : \mathbb{R}^{3,1} \to \mathcal{V}_\Lambda, \quad \Lambda \in S = \bar{L}_W[\mathfrak{g}]$$

where R_Λ are irreps of $\mathfrak{g}_{\mathbb{R}}$ acting on representation space $\mathcal{V}_\Lambda$ labelled by weights Λ.

Full Lagrangian.
With strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ and the covariant derivative $\mathrm{D}_\mu = \partial_\mu + R_\Lambda(A_\mu)$,

$$\mathcal{L} = \frac{1}{g^2}\kappa(F_{\mu\nu}, F^{\mu\nu}) + \sum_{\Lambda \in S}(\mathrm{D}_\mu \phi_\Lambda, \mathrm{D}^\mu \phi_\Lambda) - W\left(\{(\phi_\Lambda, \phi_\Lambda) : \Lambda \in S\}\right)$$

is invariant under the gauge transformation

$$\delta_X \phi_\Lambda = \epsilon R_\Lambda(X)\phi_\Lambda, \quad \delta_X A_\mu = -\epsilon \partial_\mu X + \epsilon[X, A_\mu]$$

specified by $X : \mathbb{R}^{3,1} \to \mathfrak{L}(G)$.