Gravitational collapse and Swiss-Cheese cosmologies on the brane

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Status of BHs in brane-worlds

• RS2 brane-world $\rightarrow$ no BH
• Friedmann brane in SchAdS5 bulk
  $\rightarrow$ BH in the bulk, no BH on the brane
• 5d Schwarzschild $\rightarrow$ 4d section not cosmological
• AdS5 bulk $\rightarrow$ BH known only in the weak field limit, $r^{-3}$ correction
• tidal charged brane BH, $r^{-2}$ correction $\rightarrow$ bulk not known

• gravitational collapse of dust sphere on the brane
  $\rightarrow$ static exterior and KK energy density:
    bounce / black hole / naked singularity
  $\rightarrow$ no static exterior if no KK energy density
• no brane stars with vanishing surface-pressure and static exterior
• exterior can be made static by including a radiation layer
Black holes on cosmological background in GR

**Einstein-Straus Swiss-cheese model**

cut out spheres of constant comoving radius from FLRW

fill with vacuum spheres containing Schwarzschild BHs

**main features:**

- cosmic expansion does not affect planetary motions
- luminosity distance – redshift relation changed
- unstabil under perturbations
the McVittie cosmology

\[ ds^2 = - \frac{a^2}{(1+4k^2x^2/4)^{1/2}} \left( \frac{a^2}{(1+4k^2x^2/4)^{1/2}} + \frac{r_0}{xa^{1/2}} \right)^4 \left( dx^2 + x^2 d\Omega^2_2 \right) \]

- reduces to Schwarzschild for \( k=0, \ a=\text{const} \)
- to FLRW for \( r_0=0 \)
- cosmic expansion drives
  1. planetary orbits into outward spiralling
  2. galactic clusters into expansion
     (however slower than the cosmic expansion)
**Schwarzschild BH on the brane**

**Conjecture:** can be embedded into the bulk only by extending the singularity into the bulk


**black string with singular AdS horizon**

Gregory-Laflamme **instability**


**black cigar**

**BH is not localized on the brane!**
Swiss-cheese brane-worlds

Friedmann brane with Schwarzschild voids

To study:
• junctions across the brane
• junctions on the brane
Embedding the brane into the bulk

Variations in the extrinsic curvature

[Diagram showing a brane black holes and Swiss-cheese brane]

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**Model:** black strings / cigars penetrate the cosmological bulk

**Conjecture:** AdS5 and black string / cigar regions can be matched by proper choice of the transition regions
Junction on the brane

Schwarzschild metric in the voids in curvature coords.: 

\[ ds_s^2 = -(1 - \frac{2m}{R})dT^2 + \left(1 - \frac{2m}{R}\right)^{-1}dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2). \]

Brane metric (FLRW) in comoving coords.: 

\[ ds_{FLRW}^2 = -d\tau^2 + a^2(\tau)[d\chi^2 + \mathcal{H}^2(\chi; k) \times (d\theta^2 + \sin^2\theta d\varphi^2)], \]

\[ \mathcal{H}(\chi; k) = \begin{cases} 
\sin\chi, & k = 1, \\
\chi, & k = 0, \\
\sinh\chi, & k = -1. 
\end{cases} \]

generalized Friedmann and Raychaudhuri eqs. 

For the scale-factor \( a(\tau) \):

\[ \frac{\dot{a}^2 + k}{a^2} = \frac{\Lambda}{3} + \frac{\kappa^2\rho}{3}\left(1 + \frac{\rho}{2\lambda}\right), \]

\[ \frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{\kappa^2}{6}\left[\rho\left(1 + \frac{2\rho}{\lambda}\right) + 3p\left(1 + \frac{\rho}{\lambda}\right)\right] \]
Continuity of the induced metric

Induced metrics at constant comoving radius:

\[ ds^2_{\text{int}} = \left[ -\left(1 - \frac{2m}{R_0}\right)T_0^2 + \left(1 - \frac{2m}{R_0}\right)^{-1}R_0^2 \right] d\tau^2 + R_0^2(d\theta^2 + \sin^2\theta d\varphi^2), \]

\[ ds^2_{\text{ext}} = -d\tau^2 + a^2(\tau)\left[ \mathcal{H}_0^2(d\theta^2 + \sin^2\theta d\varphi^2) \right], \]

where \( R_0 = R(\tau, \chi_0), T_0 = T(\tau, \chi_0), \mathcal{H}_0 = \mathcal{H}(\chi_0; k) \)

Their continuity gives the motion of the boundary:

\[ R_0 = a(\tau)\mathcal{H}_0. \]

\[ \left(1 - \frac{2m}{a(\tau)\mathcal{H}_0}\right)^2 T_0^2 = 1 - \frac{2m}{a(\tau)\mathcal{H}_0} + \dot{a}^2(\tau)\mathcal{H}_0^2. \]

Same eqs. as in G.R., modified cosmological dynamics for \( a(\tau) \)

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Continuity of the extrinsic curvature

Non-trivial components of the extrinsic curvature:

\[ K^{\text{int}}_{\theta\theta} = \left(1 - \frac{2m}{R_0}\right) R_0 \dot{T}_0, \quad K^{\text{int}}_{\phi\phi} = K_{\theta\theta} \sin^2 \theta. \]

\[ K^{\text{ext}}_{\theta\theta} = a(\tau) \mathcal{H}_0 (1 - k \mathcal{H}_0^2)^{1/2}, \quad K^{\text{ext}}_{\phi\phi} = K_{\theta\theta} \sin^2 \theta, \quad K^{\text{ext}}_{\tau\tau} = \ldots \]

Their continuity implies:

\[ \dot{T}_0 = \left(1 - \frac{2m}{a(\tau) \mathcal{H}_0}\right)^{-1} (1 - k \mathcal{H}_0^2)^{1/2}, \quad \dot{a}^2(\tau) + k = \frac{2m}{a^2(\tau) \mathcal{H}_0^3}. \]

and:

\[ \ddot{a}(\tau) = -\frac{m}{a^2(\tau) \mathcal{H}_0^3}. \]

LHS given by the generalized Friedmann and Raychaudhuri eqs.
Comparison with GR

**Junction conditions + dynamics:**

\[
m = \left[ \Lambda + \kappa^2 \rho(\tau) \left( 1 + \frac{\rho(\tau)}{2\Lambda} \right) \right] \frac{a^3(\tau) \mathcal{H}^3_0}{6},
\]

\[
p(\tau) = \frac{\Lambda}{\kappa^2 (1 + \frac{\rho(\tau)}{\Lambda})} - \frac{\rho(\tau)^2}{2[\rho(\tau) + \Lambda]}.
\]

**EOS of the cosmological fluid**

**Differences:**

1. The cosmological fluid is not dust, even when \( \Lambda = 0 \)

2. The difference is significant in the high energy / early universe regime.

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Imposing $\rho = \text{const}$

The relation between the BH mass, comoving radius of the void and the equation of state of the cosmological fluid:

\[
\begin{align*}
\alpha &= \text{const} \\
m &= 0 = \rho + \chi \\
k &= 0 \\
\rho_{1,2} &= -\lambda \pm \sqrt{\lambda (\lambda - 2\Lambda / \kappa^2)}
\end{align*}
\]

The fluid cancels the contribution of Minkowski brane
The continuity equation

\[ \dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) = 0 \]

integrated gives

\[ a^3 = C \frac{\kappa^2}{2\lambda} \left[ \Lambda + \kappa^2 \rho \left( 1 + \frac{\rho}{2\lambda} \right) \right]^{-1} \]

Inserted into the mass – density – comoving radius relation, gives the mass:

\[ m = \frac{\kappa^2 C}{12\lambda} \mathcal{H}_0^3. \]

Friedmann eq. integrates for k=0 to:

\[ a^3 = \frac{3\kappa^2 C}{8\lambda} \tau^2 \]

Swiss-cheese brane-world

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Dynamics on the brane

Junction conditions on the brane: no static Swiss-cheese universes

Dynamical Swiss-cheese universes:

\[ a^3 = \frac{9m\tau^2}{2\chi_0^3}, \]

forever expand and decelerate

contain perfect fluid with EOS:

\[ a^3(\rho + p)(\rho + \lambda) = \frac{6m\lambda}{\kappa^2 \chi_0^3}. \]

energy density

\[ \frac{\rho_{1,2}}{\lambda} = \frac{1}{1 + \sqrt{1 - \frac{2\Lambda}{\kappa^2 \lambda} + \frac{8}{3\kappa^2 \lambda \tau^2}}}. \]

pressure for the + branch

\[ \frac{p}{\lambda} = 1 - \frac{4 + 3(\kappa^2 \lambda - 2\Lambda)\tau^2}{(3\lambda)^{1/2} \kappa \tau \sqrt{8 + 3(\kappa^2 \lambda - 2\Lambda)\tau^2}}. \]
Domains of well-definedness

\[ \Lambda = \text{difference of the cosmological constants in the Friedmann and Schwarzschild regions} \]

Can be transformed away into the fluid variables →

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \tau &lt; \tau_1 )</th>
<th>( \tau = \tau_1 )</th>
<th>( \tau_1 &lt; \tau \leq \tau_2 )</th>
<th>( \tau &gt; \tau_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Lambda \leq 0 )</td>
<td>+</td>
<td>+</td>
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<td>+</td>
</tr>
<tr>
<td>( 0 &lt; \Lambda \leq \frac{\kappa^2 \Lambda}{2} )</td>
<td>+</td>
<td>0</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>( \Lambda &gt; \frac{\kappa^2 \Lambda}{2} )</td>
<td>+</td>
<td>0</td>
<td>–</td>
<td>no real solution</td>
</tr>
</tbody>
</table>

\[
\frac{\bar{\rho}}{\lambda} = -1 + \sqrt{1 + \frac{2 \Lambda}{\kappa^2 \lambda} + \frac{2 \rho}{\lambda} \left( 1 + \frac{\rho}{2 \lambda} \right)}.
\]

\[
\frac{\bar{p}}{\lambda} = 1 - \frac{1 + \frac{2 \Lambda}{\kappa^2 \lambda} + \frac{\rho}{\lambda} - \frac{\rho}{\lambda} (1 + \frac{\rho}{2 \lambda})}{\sqrt{1 + \frac{2 \Lambda}{\kappa^2 \lambda} + \frac{2 \rho}{\lambda} (1 + \frac{\rho}{2 \lambda})}}.
\]

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Conventional 4D interpretation

Source terms in the modified Einstein eq.

\[- \Lambda g_{ab} + \kappa^2 T_{ab} + \tilde{\kappa}^4 S_{ab} = \frac{4}{3\tau^2} u_a u_b.\]

\[\kappa^2 \rho^{\text{tot}}\]

combine to a dust similar to the Einstein-Straus model
Evolution with no $\Lambda$
Evolution with small $\Lambda$

$$\Lambda = \kappa^2 \lambda / 4.$$
Evolution with huge $\Lambda$

\[ \Lambda = \kappa^2 \lambda. \]

Pressure singularity!
Cosmic singularities

Standard cosmological model, $k=1$

**Big Bang**    $a \rightarrow 0$

and

**Big Crunch**  $a \rightarrow 0$

Dark energy models:

**BB**    $d^2a/dt^2 \rightarrow \infty$, $p \rightarrow \infty$

**Big Rip**   $H \rightarrow \infty$, $a$ regular

**BB**    Sudden future singularity

**BB**    $d^2a/dt^2$ regular, $p \rightarrow \infty$ !!!!

Swiss-cheese brane-universes, $k=0$, huge $\Lambda$ or asymmetry:

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Effects of asymmetric embedding


\[ \alpha = \frac{3 \left( \Delta \tilde{\Lambda} \right)^2}{8 \kappa^2 \lambda^3} > 0 \]

Same cosmological evolution

Different evolution of the fluid variables

→ more branches
→ occurrence of the pressure singularity more likely

\[ \alpha_{\text{crit}} = \left( \frac{1}{2} - \frac{\Lambda}{\kappa^2 \lambda} \right)^2 \]

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Effects of asymmetric embedding II.

\( \alpha = \text{asymmetry parameter} \)

\( \alpha = 0.81, \quad \Lambda = -1 \)

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Pressure singularity due entirely to the asymmetry !!!
Asymmetry speeds up the occurrence of pressure singularity!!!
Gravitational collapse on the brane


**Stellar model:** perfect fluid, Friedmann metric, boundary in free fall (but no $p=0$ there!), no KK energy density

**Static exterior:** Schwarzschild

→ same junction conditions as for the Swiss-cheese universes

**Evolution of the scale factor of the star**

\[ \frac{a^{3/2}}{a_0^{3/2}} = 1 - \left( \frac{9m}{2\chi_0^3} \right)^{1/2} \tau \]

**Relation between Schwarzschild mass $m$ and physical mass $M$**

\[ m = M \left( 1 + \frac{\rho}{2\lambda} \right) \]

\[ m = \frac{4\pi a^3 \chi_0^3 \rho}{3} \left( 1 + \frac{\rho}{2\lambda} \right) \]
Fluid evolution

**EOS:**

\[
p_{\pm} = \frac{1}{2} \left( 1 - \frac{\rho_{\pm}}{\lambda} \right) - \frac{1}{2} \left( 1 + \frac{\rho_{\pm}}{\lambda} \right)^{-1}
\]

**Energy density:**

\[
\frac{\rho_{\pm}}{\lambda} = -1 \pm \sqrt{1 + \frac{3m}{2\pi \lambda \chi_0^3 \left( a_0^{3/2} - \left( \frac{9m}{2\chi_0^3} \right)^{1/2} \tau \right)^2}}
\]

**Pressure:**

\[
\frac{p_{\pm}}{\lambda} = 1 + \frac{1}{2} \sqrt{1 + \frac{3m}{2\pi \lambda \chi_0^3 \left( a_0^{3/2} - \left( \frac{9m}{2\chi_0^3} \right)^{1/2} \tau \right)^2}} + \frac{1}{2} \sqrt{1 + \frac{3m}{2\pi \lambda \chi_0^3 \left( a_0^{3/2} - \left( \frac{9m}{2\chi_0^3} \right)^{1/2} \tau \right)^2}}
\]

Low-energy (initial stage) of the collapse:

\[p_{\pm} \approx -\frac{\rho_{\pm}^2}{2\lambda}\]

basically dust

High-energy (final stage) of the collapse:

\[p_{\pm} \approx -\frac{\rho_{\pm}}{2}\]

dark energy
Fluid evolution II.

Energy density:

Pressure:
Summary and Outlook

• Dynamical Swiss-cheese brane-worlds possible
  → if Λ different in the two regions, new type of pressure singularity

• Asymmetry makes pressure singularities more generic
  → for any Λ a critical value of the asymmetry parameter

• Gravitational collapse of perfect fluid (rather than dust) can occur in a static exterior (even without KK energy density)
  → dark energy is produced
  → but will not stop gravitational collapse, due to the $T^2$ terms

Future prospects: include KK energy density